What is a Polytope?

Any standard text in polyhedral geometry will tell you one (or both) of the following definitions, usually preceded or followed by a chunk of linear algebra explaining what these words actually mean.

1. A d-polytope P is the *convex hull* of finitely many points in $\mathbb{R}^d$.

2. A d-polytope P is the bounded intersection of finitely many *half-spaces* in $\mathbb{R}^d$.

However, since polytopes are supposed to be geometric objects, let's look at pictures instead, and go from there. We'll focus on convex hulls.

1 Convex Hulls

Intuitively, the *convex hull* captures the idea of points being inside other points. Imagine placing three thumbtacks on a board. Now wrap a piece of string around the thumbtacks until you return to where you started. The region now enclosed by the string is the convex hull of your three thumbtacks. Here's an example:

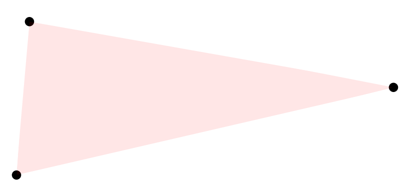

Figure 1: The convex hull (shaded) of the 3 black points

The shaded triangle is the convex hull of the three black points. Now start with as many thumbtacks as you like, and wrap a string around the outside. Again, the region enclosed by the string is the convex hull of your thumbtacks. Another way to construct a polytope is by wrapping a rubber band around your points and letting it shrink. The enclosed region is again your polytope.

Here's an example with more points:

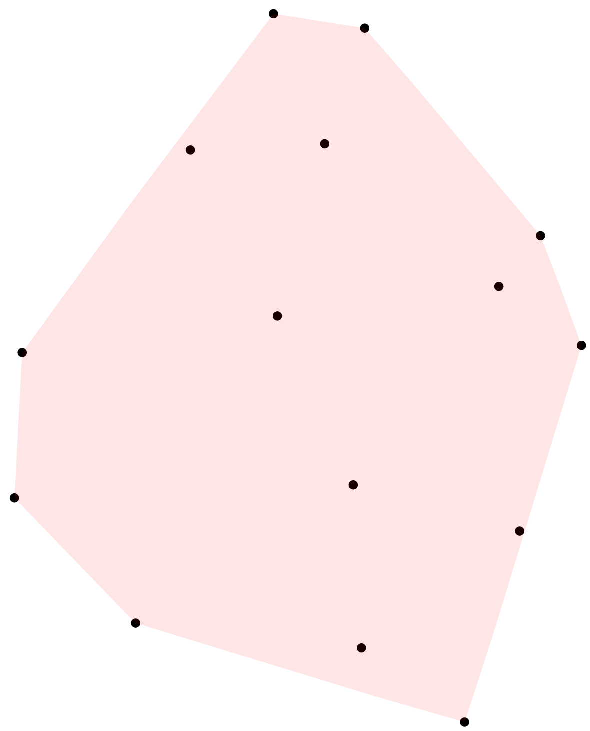

Figure 2: The convex hull (shaded) of the 15 black points

Here, we have a polytope with more sides than before, and some points landed inside the shaded region. This means that they were not necessary to define the polytope.

Now we can keep going. Instead of just adding more points however, we'll add more dimensions too. In higher dimensions, imagine picking some points, and then putting a stretchy rubber suit around those points. Eventually, as the rubber shrinks aruond your points, you'll see the convex hull as the region enclosed by the rubber suit. For example, the 3-cube is the convex hull of the points: $(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (0, 1, 1), (1, 1, 1)$.

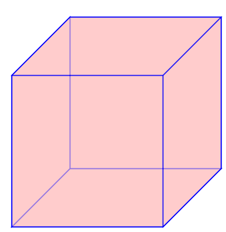

Figure 3: The 3-cube is the convex hull of its corners

2 Faces

Now that we have a rough idea about polytopes, we can study some of their properties. A textbook definition of a *face* of a polytope goes as follows: A *face* of a d-polytope P is a subset $F \subset P$ maximized by some linear functional on $\mathbb{R}^d$. However, as before, we can define a face much more geometrically.

Imagine dropping a cube onto a flat surface. The points of the cube that touch the ground form a *face* of the cube. Most likely, the cube will fall on one of its 8 square sides, so these squares are faces of the cube. Less likely, the cube could have fallen on one of its 12 edges, so these edges are also faces of the cube. Least likely, the cube could have fallen onto one of its corners, so these corners are also faces of the cube.

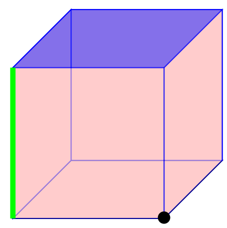

Figure 4: The blue square, green edge, and black vertex are all faces of the 3-cube

More generally, for any polytope, faces can be thought of as the sets of points that can rest on a flat surface. For example, no pair of opposite sides of a cube form a face, because there is no way the top and bottom of a cube can be placed simultaneously on the same surface.

3 Why do we care?

These abstract definitions may seem pointless at first, but polytopes turn out to be very useful tools in real life. If you've ever solved a linear programming problem (things like maximizing profit subject to some linear constraints), then you've worked with polytopes: the feasible regions of linear programs are polytopes!

Many people who study computer graphics also care about polytopes, and in particular polygons. They use polygons to model 3D-objects, and studying how to do this most efficiently is a great application of polytopes.

Polytopes are also beautiful objects in their own right. They arise in many different areas of mathematics and often give deep connections between different areas of math.