1. (a) False. The true statement is:

$$((a-b) \mod m) \equiv ((a \mod m) - (b \mod m)) \mod m$$

(remember that $x \mod m$ is an integer between 0 and $m-1$.)

(b) True. By the generalized pigeonhole principle, there is at least one day with $\lceil \frac{99}{7} \rceil = 15$ birthdays.

(c) False. There are 6^3 ways.

(d) False. This is true if c and m are relatively prime.

(e) False. The sample space has 8 outcomes.

(f) False. This is true if X and Y are independent. (Note that it is possible for $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ even if X and Y are not independent, but this is not generally true.)

(g) False. For example, $(1,2)$ and $(2,3)$ are in R, but $(1,3)$ is not.

2. Many people used linearity of expectation. However, this problem asked you to prove linearity, so you can not use it here. Instead, use the sum definition of expectation:

$$\begin{aligned} E(aX+b) &= \sum_{s\in S}(aX+b)(s)p(s) = \sum_{s\in S}(aX(s)+b)p(s) \\ &= \sum_{s\in S} aX(s)p(s) + bp(s) = \sum_{s\in S} aX(s)p(s) + \sum_{s\in S} bp(s) \\ &= a\sum_{s\in S} X(s)p(s) + b\sum_{s\in S} p(s) = aE(X) + b \end{aligned}$$

and the last line comes from $\sum_{s\in S} p(s) = 1$ since p is a probability distribution and S is its sample space.

3. (a) Use stars and bars. We want to put 9 indistinguishable objects into 3 distinguishable boxes, so the answer is $\binom{9+3-1}{3-1} = \binom{11}{2} = 55$.

(b) We want to find the number of solutions to the equation $x_1 + x_2 + x_3 = 9$ with the constraints $x_i \geq 0, x_1 \leq 5, x_2 \geq 1$. First, deal with the lower bounds, which in this case is only $x_2 \geq 1$.

Make a new variable $y_2 = x_2 - 1$. Then $y_2 \geq 0$, and $x_1 + (x_2 - 1) + x_3 = 9 - 1$, so our problem is the same as finding the number of solutions to the equation $x_1 + y_2 + x_3 = 8$ with constraints $x_1, y_2, x_3 \geq 0$ and $x_1 \leq 5$.

Finally, handle the upper bounds. To find solutions with $x_1 \leq 5$, we will subtract the solutions where $x_1 \geq 6$ from the total number of solutions to $x_1 + y_2 + x_3 = 8$ with $x_1, y_2, x_3 \geq 0$. The total number of solutions is $\binom{8+3-1}{3-1} = \binom{10}{2} = 45$. The number of solutions where $x_1 \geq 6$ is the number of solutions to $y_1 + y_2 + x_3 = 2$, where $y_1 = x_1 - 6$, and $y_1, y_2, x_3 \geq 0$. This has $\binom{2+3-1}{3-1} = \binom{4}{2} = 6$ solutions. Thus the number of solutions to the original problem is $45 - 6 = 39$, and the probability is $\frac{39}{55}$.

4. We are going to prove this by strong induction. Let $P(n)$ be the statement that $f_n + f_{n+2} = l_{n+1}$. As always, begin with the base cases:

   When $n = 0$, $f_0 + f_2 = 0 + 1 = 1$ and $l_1 = 1$, so $P(0)$ is true.

   When $n = 1$, $f_1 + f_3 = 1 + 2 = 3$ and $l_2 = 2 + 1 = 3$, so $P(1)$ is true.

   For the inductive hypothesis, we suppose $P(0), P(1), \ldots, P(n)$ are true.

   For the inductive step, we need to show $P(n+1)$ is true. This is the statement that $f_{n+1} + f_{n+3} = l_{n+2}$. To show this, we manipulate the left hand side:

$$\begin{aligned} f_{n+1} + f_{n+3} &= (f_n + f_{n-1}) + (f_{n+2} + f_{n+1}) \\ &= (f_n + f_{n+2}) + (f_{n-1} + f_{n+1}) \\ &\overset{P(n)}{=} l_{n+1} + (f_{n-1} + f_{n+1}) \\ &\overset{P(n-1)}{=} l_{n+1} + l_n = l_{n+2} \end{aligned}$$

   so we are done.

5. Apply the Chinese remainder theorem. In our system, we have $a_1 = 2, a_2 = 3, a_3 = 3$ and $m_1 = 3, m_2 = 4, m_3 = 5$. Then $M = m_1 m_2 m_3 = 60$. First, we compute inverses:

$$\begin{aligned} \left(\frac{60}{3}\right)^{-1} &\equiv 2 \mod 3 \\ \left(\frac{60}{4}\right)^{-1} &\equiv 3 \mod 4 \\ \left(\frac{60}{5}\right)^{-1} &\equiv 3 \mod 5 \end{aligned}$$

   Then the final answer is:

$$2(2)(20) + 3(3)(15) + 3(3)(12) \mod 60 = 23$$

   so every solution is of the form $23 \mod 60$.

6. Pick one out of the six people. Since they know the five others, and each is either a friend or an enemy, then the person we picked has at least three friends or at least three enemies. WLOG, we will assume they had three friends: i, j, k.

   If i, j are friends, then $1, i, j$ are three mutual friends. Similarly if j, k are friends and if i, k are friends. Thus in any of these cases, we have found three mutual friends, so we are done. If none of these happens, then i, j, k are all mutual enemies, and we have found three mutual enemies, so we are done. In any case, there are either three mutual friends or three mutual enemies.