Relations

Let A and B be two sets. A *(binary) relation* on A and B is a subset $R \subset A \times B$. We often write aRb to mean $(a, b) \in R$. The following are properties of relations on a set S (where above $A = S$ and $B = S$ are taken to be the same set S):

1. Reflexive: $(a, a) \in R$ for all $a \in S$.
2. Symmetric: $(a, b) \in R \iff (b, a) \in R$ for all $a, b \in S$.
3. Antisymmetric: $(a, b) \in R$ and $(b, a) \in R \implies a = b$.
4. Transitive: $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$ for all $a, b, c \in S$.

Exercises

For each of the following relations, which of the above properties do they have?

1. Let R be the relation on $\mathbb{Z}^+$ defined by $R = \{(a, b) \mid a \text{ divides } b\}$.

   Reflexive, antisymmetric, and transitive

2. Let R be the relation on $\mathbb{Z}$ defined by $R = \{(a, b) \mid a \equiv b \pmod{33}\}$.

   Reflexive, symmetric, and transitive

3. Let R be the relation on $\mathbb{R}$ defined by $R = \{(a, b) \mid a < b\}$.

   Symmetric and transitive

4. Let S be the set of all convergent sequences of rational numbers. Let R be the relation on S defined by $\{(a, b) \mid \lim a = \lim b\}$.

   Reflexive, symmetric, transitive

5. Let P be the set of all propositional statements. Let R be the relation on P defined by $R = \{(a, b) \mid a \to b \text{ is true}\}$.

   Reflexive, transitive

Operations on Relations

We can build new relations using old ones. Let R_1 be a relation on A and B, and let R_2 be a relation on B and C. The *composite* of R_1 and R_2 is the relation $R_1 \circ R_2$ defined to be the set $\{(a,c) \mid \exists b \in B : (a,b) \in R_1 \text{ and } (b,c) \in R_2\}$.

Let R be a relation on a set S. The closure of R with respect to some property P is a relation R' such that $R \subset R'$, R' has the property P, and R' is the smallest such set (so for any other relation R'' also with property P, then $R' \subset R''$).

Exercises

1. For each of the relations on the previous page, what is the composition $R \circ R$?

   For all of these, $R \circ R = R$. The first, second, fourth, and fifth fall under the next problem. For the third one, use the following argument:

   If $a < b$, then $a < \frac{a+b}{2} < b$ so $(a, \frac{a+b}{2}), (\frac{a+b}{2}, b) \in R$, so $(a,b) \in R \circ R$.

   On the other hand, if $(a,c) \in R \circ R$, then $(a,b),(b,c) \in R$ for some b so by transitivity we know $(a,b) \in R$.

2. Let R be a reflexive, transitive relation on a set S. Prove that $R = R \circ R$. Formulate the converse statement. Is the converse statement true?

   Let $(a,b) \in R$. Since R is reflexive, then $(a,a) \in R$. Hence $(a,a),(a,b) \in R \implies (a,b) \in R \circ R$. On the other hand, suppose $(a,c) \in R \circ R$. Then $(a,b),(b,c) \in R$ for some b, so $(a,c) \in R$ by transitivity.

   The converse statement is that if $R \circ R = R$, then R is reflexive and transitive. This is not necessarily true, for example see the third problem on the previous side.

3. If a relation R on a set S is not antisymmetric, does it have an antisymmetric closure?

   No. If R is not antisymmetric, then there is some a, b such that $(a,b),(b,a) \in R$ and $a \neq b$. There is no way to change R to make it so that $a = b$, because this is a statement about the set S rather than the relation R.

4. Let P be the set of all propositional statements, and $R = \{(a,b) \mid a \vee b \text{ is true}\}$. This relation is neither transitive nor reflexive (why not?). What is its transitive closure? What is the reflexive closure?

   It is not reflexive because for any false statement F, we have $F \vee F$ is false. It is not transitive because if a, c are false statements and b is a true statement, then $(a,b),(b,c) \in R$, but $(a,c) \notin R$.

   The reflexive closure is $R \cup \{(a,a) \mid a \in P\}$. (This is always the reflexive closure.)

   The transitive closure is $R' = P \times P$. The containment $R' \subset P \times P$ is by definition of $R \times R$. For the other containment, let $(a,b) \in P \times P$. Then for any true statement T, we have that $(a,T),(T,b) \in R$, so (a,b) must be in R'. Thus $P \times P \subset R'$.