Relations

Let A and B be two sets. A *(binary) relation* on A and B is a subset $R \subset A \times B$. We often write aRb to mean $(a, b) \in R$. The following are properties of relations on a set S (where above $A = S$ and $B = S$ are taken to be the same set S):

1. Reflexive: $(a, a) \in R$ for all $a \in S$.
2. Symmetric: $(a, b) \in R \iff (b, a) \in R$ for all $a, b \in S$.
3. Antisymmetric: $(a, b) \in R$ and $(b, a) \in R \implies a = b$.
4. Transitive: $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$ for all $a, b, c \in S$.

Exercises

For each of the following relations, which of the above properties do they have?

1. Let R be the relation on $\mathbb{Z}^+$ defined by $R = \{(a, b) \mid a \text{ divides } b\}$.

2. Let R be the relation on $\mathbb{Z}$ defined by $R = \{(a, b) \mid a \equiv b \pmod{33}\}$.

3. Let R be the relation on $\mathbb{R}$ defined by $R = \{(a, b) \mid a < b\}$.

4. Let S be the set of all convergent sequences of rational numbers. Let R be the relation on S defined by $\{(a, b) \mid \lim a = \lim b\}$.

5. Let P be the set of all propositional statements. Let R be the relation on P defined by $R = \{(a, b) \mid a \to b\}$.

Operations on Relations

We can build new relations using old ones. Let R_1 be a relation on A and B, and let R_2 be a relation on B and C. The *composite* of R_1 and R_2 is the relation $R_1 \circ R_2$ defined to be the set $\{(a, c) \mid \exists b \in B : (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$.

Let R be a relation on a set S. The closure of R with respect to some property P is a relation R' such that $R \subset R'$, R' has the property P, and R' is the smallest such set (so for any other relation R'' also with property P, then $R' \subset R''$).

Exercises

1. For each of the relations on the previous page, what is the composition $R \circ R$?

2. Let R be a reflexive, transitive relation on a set S. Prove that $R = R \circ R$. Formulate the converse statement. Is the converse statement true?

3. If a relation R on a set S is not antisymmetric, does it have an antisymmetric closure?

4. Let P be the set of all propositional statements, and $R = \{(a, b) \mid a \vee b \text{ is true}\}$. This relation is neither transitive nor reflexive (why not?). What is its transitive closure? What is the reflexive closure?