Generating Functions

Exercises

1. What is the generating function for the sequence $\{b_i\}$ where b_i is the number of length i bitstrings? (Alternatively, the cardinality of the power set of a set of i elements.) What is the generating function for the sequence $\{a_i\}$ where a_i is the number of pairs of bitstrings whose lengths sum to i? (for example, $a_0 = 1, a_1 = 4, a_2 = 12, \dots$)

   The number of length i bitstrings is 2^n, so $b_i = 2^n$, so the generating function is

   $$B(x) = \sum_{k\geq 0} 2^k x^k = \sum_{k\geq 0}(2k)^k = \frac{1}{1-2x}$$

   A pair of bitstrings whose lengths sum to i is a pair (b, b') where b is a bitstring of length a and b' is a bitstring of length $i-a$. Therefore, there are $b_a b_{n-a}$ such bitstrings. Furthermore, any a between 0 and i is possible, so we have

   $$a_i = \sum_{a\geq 0}^{i} b_a b_{n-a}$$

   and this is exactly the coefficient of x^n in the product of generating functions $B(x)^2$, so we get the generating function $A(x) = \frac{1}{(1-2x)^2}$.

2. What is the generating function for the sequence $\{a_i\}$ where a_i is the number of solutions to $x_1 + x_2 = i$ and $x_i \geq 0$? (Challenge: what about $x_1 + \cdots + x_n = i$ and $x_i \geq 0$?)

   The number of solutions is $\binom{i+2-1}{2-1} = \binom{i+1}{1} = i + 1$. The generating function is:

   $$A(x) = \sum_{k\geq 0}(k+1)x^k = \sum_{k\geq 0}(\sum_{i=0}^{k} 1 * 1)x^k = \frac{1}{(1-x)^2}$$

   (You saw this last part in class.) For the challenge, there are $\binom{i+n-1}{n-1}$ solutions.

   $$A(x) = \sum_{k\geq 0}\binom{k+n-1}{n-1}x^k = \frac{1}{(1-x)^n}$$

   (This last part is a bit difficult to show, so don't worry about it.)

3. What is the generating function for the sequence $\{a_i\}$ where a_i is the number of i element subsets of an n element set for some fixed n?

   $$A(x) = \sum_{k\geq 0}\binom{n}{k}x^k = \sum_{k=0}^{n}\binom{n}{k}x^k = (1+x)^n$$

4. Solve the linear recurrence $a_n = 2a_{n-1} + 1$ where $a_0 = 1$.

   First, multiply both sides by x^{n-1}: $a_n x^{n-1} = 2a_{n-1}x^{n-1} + x^{n-1}$. This gives an equation for each $n \geq 1$. Then adding together all these equations gives us an equation of power series:

$$\sum_{k\geq 0} a_{k+1}x^k = \sum_{k\geq 0} 2a_k x^k + \sum_{k\geq 0} x^k$$

   Multiply both sides by x:

$$\sum_{k\geq 0} a_{k+1}x^{k+1} = 2x\sum_{k\geq 0} a_k x^k + x\sum_{k\geq 0} x^k$$

   and rearrange and solve for $A(x)$:

$$A(x) - 1 = 2xA(x) + \frac{x}{1-x} \implies A(x) = \frac{1}{(1-x)(1-2x)}$$

   Using partial fractions,

$$A(x) = \frac{2}{1-2x} - \frac{1}{1-x} = 2\sum_{k\geq 0}(2x)^k - \sum_{k\geq 0} x^k$$

   so the solution to the recurrence relation is $a_i = 2^{i+1} - 1$.

Inclusion-Exclusion

Exercises

1. Write down the inclusion-exclusion formula for $|A \cup B \cup C|$, and draw a Venn diagram to visualize this formula.

   $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

2. What happens when applying inclusion-exclusion if all the sets A_i are disjoint?

   All the intersections are $\varnothing$, so $|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$.

3. Use inclusion-exclusion to prove that the number of surjective functions with domain $A = \{1, \ldots, k\}$ and codomain $B = \{1, \ldots, n\}$ is $\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-j)^k$.

   (Hint: consider the sets S of all functions from A to B and S_i all functions from A to B which do not map anything to i. Try to mimic the style of proof used to compute the number of derangements.)

   In this setup, we want to calculate $|S| - |S_1 \cup \cdots \cup S_n|$. We will use inclusion exclusion to compute $|S_1 \cup \cdots \cup S_n|$:

$$|S_1 \cup \cdots \cup S_n| = \sum_{i=1}^{n} |S_i| - \sum_{1\leq i<j\leq n} |S_i \cap S_j| + \cdots + (-1)^{n-1}|S_1 \cap \cdots \cap S_n|$$

Note that $|S_{i_1} \cap \cdots \cap S_{i_m}| = (n-m)^k$ because we have $n-m$ choices for each of the k elements of A. Plugging in and simplifying gives the desired formula.

$$\begin{aligned}
|S| - |S_1 \cup \cdots \cup S_n| &= n^k - \left(\sum_{i=1}^{n} |S_i| - \sum_{1 \le i < j \le n} |S_i \cap S_j| + \cdots + (-1)^{n-1} |S_1 \cap \cdots \cap S_n| \right) \\
&= (-1)^0 \binom{n}{0} (n-0)^k + (-1)^1 \sum_{i=1}^{n} (n-1)^k + (-1)^2 \sum_{1 \le i < j \le n} (n-2)^k \\
&+ \cdots + (-1)^n \binom{n}{n} (n-n)^k \\
&= (-1)^0 \binom{n}{0} (n-0)^k + (-1)^1 \binom{n}{1} (n-1)^k + (-1)^2 \binom{n}{2} (n-2)^k \\
&+ \cdots + (-1)^n \binom{n}{n} (n-n)^k \\
&= \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k
\end{aligned}$$