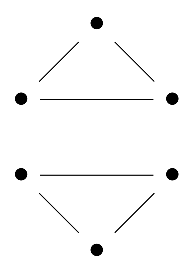
An *Euler path* in a graph G is a simple path (no repeated edges) containing every edge of G. An *Euler circuit* is an Euler path beginning and ending at the same vertex. We have two theorems about when these exist:

1. A connected graph G with at least 2 vertices has an Euler circuit iff each vertex has even degree.
2. A connected graph G hass an Euler path that is not an Euler circuit iff it has exactly two vertices of odd degree (and all other vertices have even degree).

Exercises

1. Why are these theorems false if G is not connected?

   Here's a counterexample if G is not connected, even though every vertex has even degree:

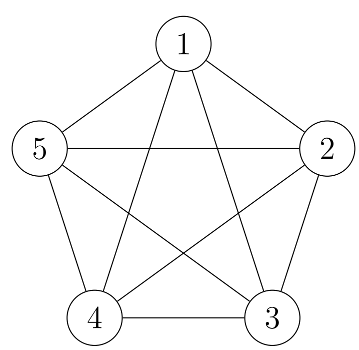
   

2. Show that having an Eulerian circuit (or path) is a graph invariant.

   Let $G = (V_1, E_1)$ be isomorphic to $H = (V_2, E_2)$ by some isomorphism $f : V_1 \to V_2$, and suppose $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ is an Euler path in G. Then apply f to this Euler path to get $(f(x_1), f(x_2)), (f(x_2), f(x_3)), \ldots, (f(x_{n-1}), f(x_n))$, which you can (and should) check is an Euler path in H.

   Another way to say this is that if all the degrees of vertices in G are even, then they must also be all even in H.

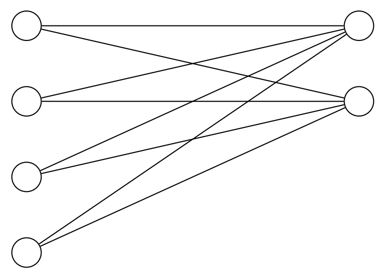
3. Which complete graphs K_n have Eulerian circuits? Find an Eulerian circuit in K_5:

   K_n has an Eulerian circuit if n is odd. Here's one possible Eulerian circuit in K_5:

   $(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (1, 3), (3, 5), (5, 2), (2, 4), (4, 1)$ (go around the outside, and then go around the star).

4. Find an Eulerian circuit in the following graph:



Label the left vertices $1, 2, 3, 4$ (in order going down) and the right vertices $5, 6$ (in order going down). Then an Eulerian circuit is:

$(5, 1), (1, 6), (6, 2), (2, 5), (5, 3), (3, 6), (6, 4), (4, 5)$.

5. If $G = (V, E_1)$ and $H = (V, E_2)$ have Eulerian circuits, which of the following also have Eulerian circuits? (Hint: use the theorems about the degree. What can the degrees of the vertices be in each of the following situations?)

   (a) $G_1 = (V, E_1 \cup E_2)$

   G_1 does not necessarily have an Eulerian circuit.

   For example, the following graphs:

   have union equal to K_4, which does not have an Eulerian circuit, because all vertices have odd degree, even though both graphs have Eulerian circuits.

   (b) $G_2 = (V, E_1 \cap E_2)$

   Using the same example as above, we can see that the intersection is not even connected, so the intersection does not necessarily have an Eulerian circuit either.

   (c) $G_3 = (V, E_1 \oplus E_2)$, assuming G_3 is connected. (remember $A \oplus B = (A \backslash B) \cup (B \backslash A)$ is all the elements in A or B, but not in both.)

   We need to check that every vertex in G_3 has even degree. Let $v \in V$. Let $a_1, \ldots, a_n$ be the edges adjacent to v in G, and $b_1, \ldots, b_m$ be the edges adjacent to v in H. We know that n and m are even because G and H have Eulerian circuits.

   Then the possible adjacent edges to v in G_3 are $a_1, \ldots, a_n, b_1, \ldots, b_m$. For each edge that is in both G and H, we remove one a_i and one b_j from the list. Thus for each edge in the intersection, we remove two edges from this list. Thus there are $n + m -$ (even number) of edges adjacent to v in G_3. Since n and m were even, this number is even, so every vertex in G_3 has even degree, so G_3 has an Eulerian circuit.