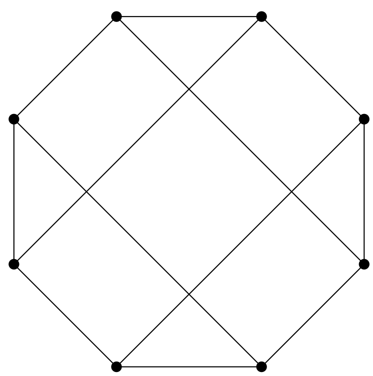
Graph Isomorphisms

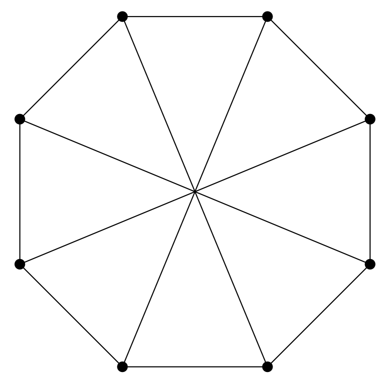
Exercises

1. Show that being bipartite is a graph invariant. (Let G and H be isomorphic graphs, and suppose G is bipartite. Then show that H is also bipartite.)

   Let $G = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isomorphic graphs, so there is a bijection $f : V_1 \to V_2$ such that $(a, b) \in E_1 \iff (f(a), f(b)) \in E_2$. Since G is bipartite, there is some bipartition $V_1 = A \cup B$. Then $V_2 = f(A) \cup f(B)$ is a bipartition of H. (If there was an edge $(f(a), f(b)) \in f(A)$, then we would have an edge $(a, b) \in A$, contradicting that $A \cup B$ was a bipartition. Similarly for B.)

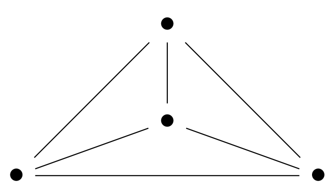
2. Use the previous problem to show that the following graphs are not isomorphic:

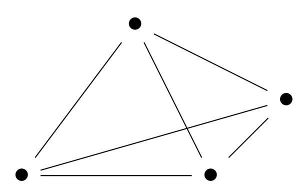
   

   The right graph has a cycle of length 5, so is not bipartite, while the graph on the left is bipartite (for the bipartition, take alternating vertices around the outside).

3. Show that the following two graphs are isomorphic, and furthermore that any bijection of the respective vertex sets is actually an isomorphism.

   Just write it out and verify.

4. (Challenge) More generally, show that K_n is isomorphic to itself via any bijection on the vertices.

   Here is the outline of a proof (don't worry about this problem too much):

   Let $K_n = (\{1, 2, \ldots, n\}, E)$, where E is all possible $\binom{n}{2}$ edges. Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a bijection. Then to check that f is an isomorphism, we need to verify that $(a, b) \in E \iff (f(a), f(b)) \in E$. However, since every possible edge is in E, then whatever a, b are, we always have $(a, b) \in E$ and $(f(a), f(b)) \in E$, so f is an isomorphism.

Connectivity

Exercises

1. Show that connectedness is a graph invariant.

   Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isomorphic graphs with a bijection $f : V_1 \to V_2$, and suppose that G_1 is connected. Then for any $a, b \in V_2$, there are $a', b' \in V_1$ such that $f(a') = a$ and $f(b') = b$ since f is a bijection. Since G_1 is connected, there is a sequence of edges $(a', x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, b')$ from a' to b'. Then applying f gives a sequence of edges $(a, f(x_1)), (f(x_1), f(x_2)), \ldots, (f(x_{n-1}), f(x_n)), (f(x_n), b)$ from a to b in G_2. Thus any pair of vertices in G_2 is connected by a path, so G_2 is connected.

2. Let $C(x) = \sum_{n\geq 0} c_n x^n$ be the generating function for the number of nonisomorphic, connected, simple graphs. (By convention, $c_0 = 0$.) Let $G(x) = \sum_{n\geq 0} c_n x^n$ be the generating function for simple graphs with exactly 2 connected components. Show that $G(x) = \frac{1}{2}C(x)^2$.

   Recall the definition of multiplying generating functions:

   $$\frac{1}{2}C(x)^2 = \sum_{n\geq 0} \frac{1}{2}\left(\sum_{i=0}^{n} c_i c_{n-i}\right) x^n$$

   For each n, a simple graph with exactly two connected components is a partition of the n vertices into two sets, and a connected graph on each set. The number of ways to do this is exactly the coefficient of x^n above. We need the one half because otherwise we double count many graphs. (i.e. when we partition into i and $n-i$, we could instead have partitioned into $n-i$ and i, and these graphs will be isomorphic.)

3. What is the minimum number of edges for a graph G needed to be connected? What is the maximum number of edges a graph can have without being connected?

   The minimum number of edges to be connected is $n-1$, where n is the number of vertices. The maximum number of edges to be disconnected is $\binom{n-1}{2}$.