Graph Isomorphisms

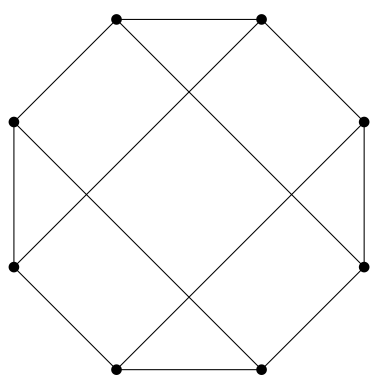
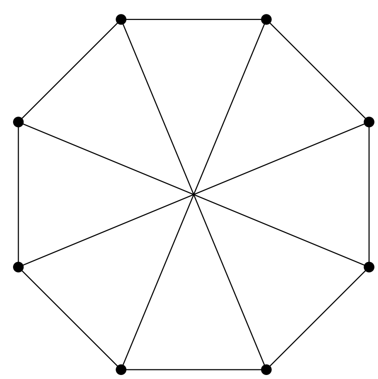
Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $f : V_1 \to V_2$ such that $(a, b) \in E_1 \iff (f(a), f(b)) \in E_2$.

A *graph invariant* is a property of a graph that is preserved by isomorphisms. (If graphs G_1 and G_2 are isomorphic, and G_1 has some invariant property, then G_2 must have the same property.) Common examples of graph invariants are the number of edges, the number of vertices, the degree of a vertex, and there are many others.

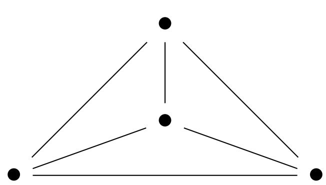
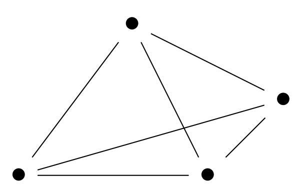
Exercises

1. Show that being bipartite is a graph invariant. (Let G and H be isomorphic graphs, and suppose G is bipartite. Then show that H is also bipartite.)

2. Use the previous problem to show that the following graphs are not isomorphic:

3. Show that the following two graphs are isomorphic, and furthermore that any bijection of the respective vertex sets is actually an isomorphism.

4. (Challenge) More generally, show that K_n is isomorphic to itself via any bijection on the vertices.

Connectivity

A *path* of length n in a graph is a sequence of edges $(e_1, e_2, \ldots, e_n)$ where $e_i = (x_i, x_{i+1})$. A *cycle* is a path of length ≥ 1 beginning and ending at the same vertex. A path is *simple* if its edges are distinct.

An undirected graph is *connected* if there is a path between every pair of distinct vertices. A *connected component* of a graph G is a connected subgraph of G that is not contained in any other connected subgraph of G. (In other words, it is a maximal connected subgraph.)

A directed graph is *strongly connected* if for any ordered pair (a, b) there is a path from a to b. It is *weakly connected* if for every unordered pair $\{a, b\}$, there is a path from a to b or a path from b to a.

Exercises

1. Show that connectedness is a graph invariant.

2. Let $C(x) = \sum_{n\geq 0} c_n x^n$ be the generating function for the number of connected, simple graphs. (By convention, $c_0 = 0$.) Let $G(x) = \sum_{n\geq 0} c_n x^n$ be the generating function for simple graphs with exactly 2 connected components. Show that $G(x) = \frac{1}{2}C(x)^2$.

3. What is the minimum number of edges for a graph G needed to be connected? What is the maximum number of edges a graph can have without being connected?