Generating Functions

A *generating function* is a representation of a sequence $a_0, a_1, a_2, \ldots$ as a (formal) power series $\sum_{i\geq 0} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots$. Formal means that we do not worry about any convergence issues and we never plug in any numerical values for x. This may seem pointless, but the idea is that operations on sequences can become much more natural as operations on power series instead.

Operations on Power Series

Let $f(x) = \sum_{k\geq 0} a_k x^k$ and $g(x) = \sum_{k\geq 0} b_k x^k$ be two power series. We define:

1. The sum: $f(x) + g(x) = \sum_{k\geq 0}(a_k + b_k)x^k$
2. The product: $f(x)g(x) = \sum_{k\geq 0}(\sum_{i=0}^{k} a_i b_{k-i})x^k$
3. The (formal) derivative: $f'(x) = \sum_{k\geq 0}(k+1)a_{k+1}x^k$
4. The (formal) integral: $\int f(x) = \sum_{k\geq 0} \frac{a_k}{k+1} x^{k+1}$

Exercises

1. What is the generating function for the sequence $\{b_i\}$ where b_i is the number of length i bitstrings? (Alternatively, the cardinality of the power set of a set of i elements.) What is the generating function for the sequence $\{a_i\}$ where a_i is the number of pairs of bitstrings whose lengths sum to i? (for example, $a_0 = 1, a_1 = 4, a_2 = 12, \ldots$)

2. What is the generating function for the sequence $\{a_i\}$ where a_i is the number of solutions to $x_1 + x_2 = i$ and $x_i \geq 0$? (Challenge: what about $x_1 + \cdots + x_n = i$ and $x_i \geq 0$?)

3. What is the generating function for the sequence $\{a_i\}$ where a_i is the number of i element subsets of an n element set for some fixed n?

4. Solve the linear recurrence $a_n = 2a_{n-1} + 1$ where $a_0 = 1$.

Inclusion-Exclusion

So far, we've seen the inclusion-exclusion rule for two sets: $|A \cup B| = |A| + |B| - |A \cap B|$. In English, this says: to count the number of elements in $A \cup B$, we first add the number of elements in A and the number of elements in B, and then subtract the number of elements in the intersection because we counted them twice. This trend holds more generally as well:

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|$$

The interpretation is similar to before.

Exercises

1. Write down the inclusion-exclusion formula for $|A \cup B \cup C|$, and draw a Venn diagram to visualize this formula.

2. What happens when applying inclusion-exclusion if all the sets A_i are disjoint?

3. Use inclusion-exclusion to prove that the number of surjective functions with domain $A = \{1, \ldots, k\}$ and codomain $B = \{1, \ldots, n\}$ is $\sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^k$.

   (Hint: consider the sets S of all functions from A to B and S_i all functions from A to B which do not map anything to i. Try to mimic the style of proof used to compute the number of derangements.)