Graphs

Exercises

1. How many vertices and edges does K_n have?

   There are n vertices and $\binom{n}{2}$ edges.

2. Show that K_n is not bipartite when $n \geq 3$. Show that W_n is not bipartite when $n \geq 2$.

   The triangle graph $G = (\{1, 2, 3\}, \{(1, 2), (2, 3), (1, 3)\})$ is not bipartite (check this e.g. by trying every bipartition). In each case (K_n for $n \geq 3$ or W_n for $n \geq 2$), the graph contains a triangle subgraph, so it is not bipartite. (If it were, then the restriction of the bipartition to the triangle would show that the triangle is bipartite, a contradiction.)

3. Show that the number of undirected graphs on n vertices (so $|V| = n$) is $2^{\binom{n}{2}}$.

   The total number of possible edges is $\binom{n}{2}$. We can think of an undirected graph as: for each edge, decide whether or not that edge is in the graph. Thus we have 2 choices $\binom{n}{2}$ times, so the total number of graphs is $2^{\binom{n}{2}}$.

4. For which values of $n \geq 1$ is C_n bipartite?

   C_n is bipartite when n is even, and C_n is not bipartite when n is odd. (For even n, the bipartition is $\{1, 3, \ldots, n-1\} \cup \{2, 4, \ldots, n\}$.

   Check that C_n is not bipartite when n is odd by contradiction: if it were bipartite, then there is some coloring with 2 colors such that no two vertices of the same color are adjacent. Try to find such a coloring by first choosing a vertex and considering what color it can be.

5. Let R be a relation from A to B. Explain how we may think of R as a directed, bipartite graph.

   Since R is a relation from A to B, then $R \subset A \times B$. Then we have a directed graph $G = (A \cup B, R)$ which is bipartite with bipartition $A \cup B$.

6. Let R_1 be a relation from A to B, and R_2 a relation from B to C. What is the directed, bipartite graph corresponding to the composition $R_2 \circ R_1$? (i.e. how do we describe it in terms of the graphs of R_1 and R_2?)

   The graph corresponding to the composition is the graph G with vertices $A \cup C$ (this is also the bipartition) and edges (a, c) whenever there is $b \in B$ such an edge $(a, b) \in G_1$ and $(b, c) \in G_2$.

7. Let R be a relation on a set A. Explain how we may think of R as a directed graph. If R is symmetric, explain how to think of R as an undirected graph.

   $G = (A, R)$ is a directed graph. If R is symmetric, then $(a_1, a_2) \in R \iff (a_2, a_1 \in R)$. Thus for each pair of vertices where we have an edge, we have a pair of opposite edges. Therefore it's the same as thinking of an undirected graph.

8. Let $G = (V, E)$ be an undirected graph. Explain how we may think of E as a relation on V. How can we tell whether this relation is transitive?

   By definition, $E \subset V \times V$, so E is already a relation on V. This relation is transitive when the "is connected" relation on G is transitive. In other words, E is transitive if for every path $a \to b$ in G, there is an edge (a, b).