Graphs

There's a lot of terminology to keep track of:

Terminology

1. Undirected Graph: A pair $G = (V, E)$, where $E \subset V \times V$. In this case, the elements (a, b) and (b, a) correspond to the same edge between a and b.
2. Directed Graph: A pair $G = (V, E)$, where $E \subset V \times V$. In this case, the elements (a, b) and (b, a) correspond to *different* edges between a and b.
3. Vertex: elements of V
4. Edge: elements of E, the two vertices connected by an edge are called its *endpoints*
5. Loop: an edge (v, v) for some $v \in V$
6. Multiple Edge: several edges with the same endpoints
7. Degree of a vertex v: in an undirected graph, this is the number of edges which have one endpoint equal to v
8. Complete graph: K_n has V an n-element set, and $E = \{(v_1, v_2) \mid v_1 \neq v_2\}$. (So we have all edges except the loops).
9. Cycle: C_n has vertices $\{v_1, \ldots, v_n\}$ and edges $(v_i, v_{i+1 \mod n})$ for $1 \leq i \leq n$.
10. Wheel: W_n is C_n with one vertex v' connected to each of the n vertices in C_n.
11. n-cube: Q_n has $V = \{\text{length } n \text{ bitstrings}\}$, and an edge between two vertices when the corresponding bitstring differ in exactly one position.
12. Bipartite Graph: Let $G = (V, E)$ be an undirected graph. G is bipartite if we can partition $V = V_1 \cup V_2$ where $V_1 \cap V_2 = \varnothing$, and $E \subset V_1 \times V_2$.

Exercises

1. How many vertices and edges does K_n have?

2. Show that K_n is not bipartite when $n \geq 3$. Show that W_n is not bipartite when $n \geq 2$.

3. Show that the number of undirected graphs on n vertices (so $|V| = n$) is $2^{\binom{n}{2}}$.

4. For which values of $n \geq 1$ is C_n bipartite?

5. Let R be a relation from A to B. Explain how we may think of R as a directed, bipartite graph.

6. Let R_1 be a relation from A to B, and R_2 a relation from B to C. What is the directed, bipartite graph corresponding to the composition $R_2 \circ R_1$? (i.e. how do we describe it in terms of the graphs of R_1 and R_2?)

7. Let R be a relation on a set A. Explain how we may think of R as a directed graph. If R is symmetric, explain how to think of R as an undirected graph.

8. Let $G = (V, E)$ be an undirected graph. Explain how we may think of E as a relation on V. How can we tell whether this relation is transitive?