Relations

A *binary relation* A to B is a subset $R \subseteq A \times B$. A relation on one set S is a binary relation where $A = B = S$. A relation R on S can have the following properties:

1. reflexive: $(x, x) \in R$ for all $x \in R$
2. symmetric: $(x, y) \in R \implies (y, x) \in R$
3. transitive: $(x, y), (y, z) \in R \implies (x, z) \in R$
4. antisymmetric: $(x, y), (y, x) \in R \implies x = y$.

Every relation has a transitive closure. (A *transitive closure* of a relation R is the smallest transitive relation containing R.) This is $R' = \{(x, y) \mid \exists n : (x, y) \in R^n\}$. In English, this is is all pairs (x, y) such that there is a chain $(x, x_1), (x_1, x_2), \ldots, (x_n, y)$ of length n in R.

Exercises

Determine whether the following are transitive and if they are not, find the transitive closure.

1. Let R be the relation on $\mathbb{Z}$ defined by $R = \{(n, n+1) \mid n \in \mathbb{Z}\}$.

   This is not transitive. For example, $(1, 2), (2, 3) \in R$, but $(1, 3) \notin R$. The transitive closure is $R' = \{(x, y) \mid x < y\}$. (This is just the less than relation.)

2. Let R be the relation on $\mathbb{Z}$ defined by $R = \{(x, y) \mid x - y = \pm 13\}$.

   This is not transitive. For example. $(0, 13), (13, 26) \in R$, but $(0, 26) \notin R$. The transitive closure is $R' = \{(x, y) \mid x \equiv y \mod 13\}$.

3. Let R be the relation on $\mathbb{Q}$ defined by $R = \{(x, y) \mid xy = 1\}$.

   This is not transitive. For example, $(\frac{1}{2}, 2), (2, \frac{1}{2}) \in R$, but $(\frac{1}{2}, \frac{1}{2}) \notin R$. The transitive closure is $R' = R \cup \{(x, x) \mid x \neq 0 \in \mathbb{Q}\}$.

Equivalence Relations

A relation on a set S is an *equivalence relation* if it is reflexive, symmetrix, and transitive. When $(x, y) \in R$, we say x is equivalent to y. For any $x \in S$, $[x]_R = \{y \in S \mid (x, y) \in R\}$ is the *equivalence class* of x. Here are important properties of equivalence classes:

1. $(x, y) \in R \iff [x]_R = [y]_R \iff [x]_R \cap [y]_R \neq \varnothing$
2. $\cup_{s \in S}[s]_R = S$.
3. Either $[x]_R \cap [y]_R = \varnothing$ or $[x]_R = [y]_R$.

Exercises

1. Which of the transitive closures on the previous side are equivalence relations? For the ones that are, what are its equivalence classes?

   The first one is not symmetric, so it is not an equivalence relation.

   The second is an equivalence relation. The eq. classes are $[x] = \{x + 13n \mid n \in \mathbb{Z}\}$. (i.e. they're all the integers congruent to a fixed integer mod 13.)

   The third is not an equivalence relation, because $(0, 0) \notin R$.

2. We defined $[x]_R$ as $\{y \in S \mid (x, y) \in R\}$. Show that this is equivalent to the alternate definition $[x]_R = \{y \in S \mid (y, x) \in R\}$.

   Since R is symmetric, we know $(x, y) \in R \iff (y, x) \in R$, so these definitions are equivalent.

3. Find the error with the following 'proof' that a symmetric, transitive relation is reflexive:

   Let R be a symmetric, transitive relation on S, and let $x \in S$. Then $(x, y) \in R$, so since R is symmetric, we have $(y, x) \in R$. Then because R is transitive, and $(x, y), (y, x) \in R$, we also have $(x, x) \in R$, so R is reflexive.

   This proof is incorrect, because we are assuming that for al $x \in S$, there is some $y \in S$ such that $(x, y) \in R$, and this is not necessarily true. For example, the last problem on the previous side was symmetric and transitive, but not reflexive, because $0 * x = 0$ for any $x \in \mathbb{Q}$.

   Give a counterexample to show that this statement is actually false.

   The last problem on the previous side is a counterexample.