Relations

A *binary relation* A to B is a subset $R \subseteq A \times B$. A relation on one set S is a binary relation where $A = B = S$. A relation R on S can have the following properties:

1. reflexive: $(x, x) \in R$ for all $x \in R$
2. symmetric: $(x, y) \in R \implies (y, x) \in R$
3. transitive: $(x, y), (y, z) \in R \implies (x, z) \in R$
4. antisymmetric: $(x, y), (y, x) \in R \implies x = y$.

Every relation has a transitive closure. (A *transitive closure* of a relation R is the smallest transitive relation containing R.) This is $R' = \{(x, y) \mid \exists n : (x, y) \in R^n\}$. In English, this is is all pairs (x, y) such that there is a chain $(x, x_1), (x_1, x_2), \ldots, (x_n, y)$ of length n in R.

Exercises

Determine whether the following are transitive and if they are not, find the transitive closure.

1. Let R be the relation on $\mathbb{Z}$ defined by $R = \{(n, n + 1) \mid n \in \mathbb{Z}\}$.

2. Let R be the relation on $\mathbb{Z}$ defined by $R = \{(x, y) \mid x - y = \pm 13\}$.

3. Let R be the relation on $\mathbb{Q}$ defined by $R = \{(x, y) \mid xy = 1\}$.

Equivalence Relations

A relation on a set S is an *equivalence relation* if it is reflexive, symmetrix, and transitive. When $(x, y) \in R$, we say x is equivalent to y. For any $x \in S$, $[x]_R = \{y \in S \mid (x, y) \in R\}$ is the *equivalence class* of x. Here are important properties of equivalence classes:

1. $(x, y) \in R \iff [x]_R = [y]_R \iff [x]_R \cap [y]_R \neq \varnothing$
2. $\cup_{s \in S} [s]_R = S$.
3. Either $[x]_R \cap [y]_R = \varnothing$ or $[x]_R = [y]_R$.

Exercises

1. Which of the transitive closures on the previous side are equivalence relations? For the ones that are, what are its equivalence classes?

2. We defined $[x]_R$ as $\{y \in S \mid (x, y) \in R\}$. Show that this is equivalent to the alternate definition $[x]_R = \{y \in S \mid (y, x) \in R\}$.

3. Find the error with the following 'proof' that a symmetric, transitive relation is reflexive:

   Let R be a symmetric, transitive relation on S, and let $x \in S$. Then $(x, y) \in R$, so since R is symmetric, we have $(y, x) \in R$. Then because R is transitive, and $(x, y), (y, x) \in R$, we also have $(x, x) \in R$, so R is reflexive.

   Give a counterexample to show that this statement is actually false.