Generalized Permutations

1. How many permutations are there of the word 'COMBINATORICS'?

   $\frac{13!}{2!2!2!}$

2. How many integers can be formed using all the digits $1, 3, 3, 3, 4, 5, 9$?

   $\frac{7!}{3!}$

3. How many ways are there to line up 3 apples, 4 bananas, 5 oranges, and 6 kiwis?

   $\frac{18!}{3!4!5!6!}$

Combinations with Repetition

There's a bit more variety with these types of problems. They all boil down to the question: how many ways are there to select k elements from an n element set if repetition is allowed? (The answer is $\binom{n+k-1}{k}$.)

1. Suppose you want to buy 10 pieces of fruit. If you can choose among mangos, dragon fruit, coconuts, and kumquats, how many ways are there to choose 10 pieces of fruit?

   $\binom{13}{10}$

2. How many ways if you want to buy at least 1 of each fruit?

   $\binom{9}{6}$

3. How many ways if you don't want more than 2 coconuts?

   $\binom{13}{10} - \binom{10}{7}$

4. Restate problems $1 - 3$ using the format of finding the number of solutions of a linear equation. (i.e. something like: How many solutions does the equation $\sum_i x_i = n$ have where $a_i \leq x_i \leq b_i$?)

   Each one has the equation $x_1 + x_2 + x_3 + x_4 = 10$.

   In problem 1, we have the constraints $x_i \geq 0$

   In problem 2, we have $x_i \geq 1$.

   In problem 3, we have $x_i \geq 0$ and $x_3 \leq 2$.

5. How many ways are there to place n indistinguishable objects into k distinguishable boxes?

   $\binom{n+k-1}{n}$.

More Combinatorial Proofs

1. Prove that $\sum_{i=0}^{k} \binom{k}{i}^2 = \binom{2k}{k}$ using a combinatorial proof.

   The right hand side counts the number of ways to pick k objects from a $2k$ element set.

   First, rewrite the left hand side: $\sum_{i=0}^{k} \binom{k}{i}\binom{k}{k-1}$. This counts the number of ways to pick k elements from a $2k$ element set as follows:

   Split the $2k$ element set into two disjoint subsets of size k. Pick i elements from the first and $k - i$ elements from the second. There are $\binom{k}{i}\binom{k}{k-i}$ ways to do this (product rule). There are k possibilities for i, so the total number of ways is $\sum_{i=0}^{k} \binom{k}{i}\binom{k}{k-i}$ (sum rule).

2. Prove $\sum_{k=0}^{n} \binom{k}{c} = \binom{n+1}{c+1}$ $(n, c \in \mathbb{N})$ using both a combinatorial and an inductive proof.

   This is the same as a previous identity: $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$.

3. Prove that $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$ using a combinatorial proof.

   Let N be an n element set. The right hand side counts the number of pairs (x, S) where $x \in S \subset N$. (First pick an element $x \in N$, and then pick a subset $S \subseteq N$ containing N.)

   We can also do the following. Pick a k element subset $S \subseteq N$. There are $\binom{n}{k}$ ways to do this. From this subset, pick an element $x \in S$. There are k choices. Then there are $k\binom{n}{k}$ ways to pick a pair (x, S) where $|S| = k$ (product rule). We can pick any k for this, so there are $\sum_{k=0}^{n} k\binom{n}{k}$ ways to do this (sum rule).