Permutations and Combinations

1. If $|S| = n$, how many r permutations of S are there? What about r-combinations?

   If $n < r$, then there are 0 r-permutations and r-combinations.

   Otherwise, there are $\frac{n!}{(n-r)!}$ r-permutations and $\binom{n}{r}$ r-combinations.

2. How many permutations of 'ABCDEFG' contain both 'ABC' and 'DE' as consecutive substrings? How many permutations of 'ABCDEFG' have A before B?

   There are 4! permutations with 'ABC' and 'DE' as consecutive substrings.

   There are $\frac{7!}{2}$ permutations with A before B.

Binomial Coefficients and Binomial Theorem

Definitions

1. The number of r-combinations of a set S with $|S| = n$ is also written as $\binom{n}{r}$ and called a **binomial coefficient**.

2. The binomial coefficients $\binom{n}{r}$ for $n \geq 0$ and $r \geq 0$ are arranged in **Pascal's triangle** as follows: the n^{th} row has the n entries $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.

3. Let $n \in \mathbb{N}$. Then $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$. (Or $(x+y)^n = \sum_{i=0}^{n} x^i y^{n-i}$.) This is the **binomial theorem**.

Exercises

1. Using induction, prove that $\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$ where $n, r \in \mathbb{N}$ and $n > r$. (In class, you saw a combinatorial proof, and we'll give an algebraic one here.)

   We prove it by induction. Our base case is $n = r$. In this case, $\sum_{i=r}^{n} \binom{i}{r} = \binom{n}{n} = 1$. The right hand side is $\binom{n+1}{n+1} = 1$.

   Now assume it is true for $n > r$. We will show it for $n + 1 > r$.

   $\sum_{i=r}^{n+1} \binom{i}{r} = \sum_{i=r}^{n} \binom{i}{r} + \binom{n+1}{r} \overset{IH}{=} \binom{n+1}{r+1} + \binom{n+1}{r} = \binom{n+2}{r+1}$.

2. Prove $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

   Use the binomial theorem. $0 = (1 + (-1))^n = \sum_{k=0}^{n} 1^{n-k} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

3. Prove $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$. Can you generalize this to $\sum_{k=0}^{n} a^k b^{n-k} \binom{n}{k}$?

   Use the binomial theorem. $(1+2)^n = \sum_{k=0}^{n} 1^{n-k} 2^k \binom{n}{k} = \sum_{k=0}^{n} 2^k \binom{n}{k}$.

   The general form is $(a+b)^n$.

Combinatorial Proofs

In class, you saw Fibonacci numbers and bitstrings with no consecutive 1's. We will prove that the number of such bitstrings of length n is the $n+2^{th}$ Fibonacci number by showing they satisfy the same recurrence.

Let b_n be the number of length n bitstrings with no consecutive 1's. Let o_n be the number of length n bitstrings ending in 1 with no conecutive 1's. Let z_n be the number of length n bitstrings ending in 0 with no consecutive 1's.

1. Show that $b_n = z_n + o_n$.

   The left hand side counts the number of bitstrings of length n with no consecutive 1's.

   The right hand side also counts these, and we have just split up bitstrings into those ending with 0 and those ending with 1.

2. Show that $z_{n+1} = b_n$.

   Let Z_{n+1} be the set of bitstrings of length $n+1$ with no repeated 1's that end with 0.

   Let B_n be the set of bitstrings of length n with no repeated 1's.

   Then the function $f : Z_{n+1} \to B_n$ which sends a bitstring of length $n+1$ with no repeated 1's and ending in 0 to the substring formed by its first n digits is a bijection.

3. Show that $o_{n+1} = z_n$.

   Let O_{n+1} be the set of bitstrings of length $n+1$ with no repeated 1's that end with 1.

   The function $g : O_{n+1} \to Z_n$ which sends a bitstring of length $n+1$ with no repeated 1's and ending in 1 to the substring formed by the first n digits is a bijection.

4. Conclude that $b_{n+2} = b_{n+1} + b_n$. Show that $b_0 = f_2$ and $b_1 = f_3$. This concludes the proof, because b_n satisfies the same recurrence relation as f_{n+2}, and they have the same base cases. (If you don't like this, try using induction to prove that they must be the same sequence.)

   $b_{n+2} = o_{n+2} + z_{n+2} = z_{n+1} + z_{n+2} = b_n + b_{n+1}$.

   $b_0 = 1, b_1 = 2, b_2 = 3$ and so on. (There was a typo in the problem. It used to say 'the $n+1^{th}$ fibonacci number.') Thus $b_n = f_{n+2}$.