Permutations and Combinations

Definitions

1. An r-**permutation** of a set S is an ordered arrangement of r elements of S.

2. An r-**combination** of a set S is an unordered selection of r elements of S.

Exercises

1. If $|S| = n$, how many r permutations of S are there? What about r-combinations?

2. How many permutations of 'ABCDEFG' contain both 'ABC' and 'DE' as consecutive substrings? How many permutations of 'ABCDEFG' have A before B?

Binomial Coefficients and Binomial Theorem

Definitions

1. The number of r-combinations of a set S with $|S| = n$ is also written as $\binom{n}{r}$ and called a **binomial coefficient**.

2. The binomial coefficients $\binom{n}{r}$ for $n \geq 0$ and $r \geq 0$ are arranged in **Pascal's triangle** as follows: the n^{th} row has the n entries $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.

3. Let $n \in \mathbb{N}$. Then $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$. (Or $(x+y)^n = \sum_{i=0}^{n} x^i y^{n-i}$.) This is the **binomial theorem**.

Exercises

1. Using induction, prove that $\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$ where $n, r \in \mathbb{N}$ and $n > r$. (In class, you saw a combinatorial proof, and we'll give an algebraic one here.)

2. Prove $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

3. Prove $\sum_{k=0}^{n} 2^k\binom{n}{k} = 3^n$. Can you generalize this to $\sum_{k=0}^{n} a^k b^{n-k}\binom{n}{k}$?

Combinatorial Proofs

In class, you saw Fibonacci numbers and bitstrings with no consecutive 1's. We will prove that the number of such bitstrings of length n is the $n+1^{th}$ Fibonacci number by showing they satisfy the same recurrence.

Let b_n be the number of length n bitstrings with no consecutive 1's. Let o_n be the number of length n bitstrings ending in 1 with no conecutive 1's. Let z_n be the number of length n bitstrings ending in 0 with no consecutive 1's.

1. Show that $b_n = z_n + o_n$.

2. Show that $z_{n+1} = b_n$.

3. Show that $o_{n+1} = z_n$.

4. Conclude that $b_{n+2} = b_{n+1} + b_n$. Show that $b_0 = f_1$ and $b_1 = f_2$. This concludes the proof, because b_n satisfies the same recurrence relation as f_{n+1}, and they have the same base cases. (If you don't like this, try using induction to prove that they must be the same sequence.)