Recurrence Relations

A *recurrence relation* for the sequence $\{a_n\}$ is an equation expressing a_n in terms of the previous terms in the sequence. A sequence is a *solution* to a recurrence relation if its terms satisfy the relation.

Solving Recurrence Relations

We'll focus on linear, homogeneous recurrence relations. These are $a_n = \sum_{i=1}^{k} c_i a_{n-i}$, and we say that such a relation has degree k. We'll consider cases when the *characteristic equation* $r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$ has distinct roots, or has degree 2 and a repeated root.

1. (degree 2, repeated roots) If the characteristic equation $r^2 - c_1 r - c_2 = 0$ has a repeated root r_0, then a sequence $\{a_n\}$ is a solution of the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for all $n \geq 0$ and some constants α_1, α_2.
2. (distinct roots) If the characteristic equation $r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$, then a sequence $\{a_n\}$ is a solution of the recurrence $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ if and only if $a_n = \sum_{i=1}^{k} \alpha_i r_i^n$ for all $n \geq 0$ and some constants α_i.

The solutions above are the "general solutions" to a recurrence. If you get values for $a_1, a_2, \ldots, a_k$, then you will also need to find the "particular solution" (this means find the constants $\alpha_1, \alpha_2, \ldots, \alpha_k$.) Don't worry about proofs for these statements. You only need to remember the statement and how to apply it.

Exercises

For the following exercises, first write down the characteristic equation corresponding to the recurrence relation, then factor the polynomial, and find a solution to the recurrence.

1. $a_n = a_{n-3}$; $a_0 = 1, a_1 = 2, a_2 = 3$

   The sequence is easy to figure out (it's $(1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots)$) because it repeats. Let's do this systematically:

   The characteristic equation is $x^3 - 1 = 0$. This has roots $x = 1, \frac{-1 \pm \sqrt{3}i}{2}$, which are distinct, so the general solution is:

   $$a_n = \alpha_1 (1)^n + \alpha_2 (\frac{-1+\sqrt{3}i}{2})^n + \alpha_3 (\frac{-1-\sqrt{3}i}{2})^n$$

   The particular solution in our case (use linear algebra) is:

   $$\alpha_1 = 2, \alpha_2 = -\frac{1}{2} + \frac{i}{2\sqrt{3}}, \alpha_3 = -\frac{1}{2} - \frac{i}{2\sqrt{3}}$$

2. $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$; $a_0 = 1$, $a_1 = 1, a_2 = 1$ This sequence is also easy to figure out (it's $(1, 1, 1, \ldots)$) because it is constant. Again, let's do it systematically:

   The characteristic equation is $x^3 - 6x^2 + 11x - 6 = 0$. The roots are $x = 1, 2, 3$, which are distinct, so the general solution is:

$$a_n = \alpha_1(1)^n + \alpha_2(2)^n + \alpha_3(3)^n$$

   The particular solution in our case is:

$$\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 0$$

3. $a_n = 4a_{n-1} - 4a_{n-2}$; $a_0 = 10, a_1 = 20$

   The characteristic equation is $x^2 - 4x + 4 = 0$. There is only one root $x = 2$, so we are in the repeated roots situation and the general solution looks like:

$$a_n = \alpha_1(2)^n + \alpha_2 n(2)^n$$

   The particular solution in our case is:

$$\alpha_1 = 10, \alpha_2 = 0$$

4. $a_n = 7a_{n-1} - 14a_{n-2} + 8a_{n-3}$; $a_0 = a_1 = a_2 = 0$

5. $a_n = 7a_{n-2} - 6a_{n-3}$; $a_0 = 4, a_1 = 1, a_2 = 15$

6. $a_n = 2a_{n-1} + 3a_{n-2}$; $a_0 = 1, a_1 = 5$

I'll leave the last three as exercises. It's the same setup as before: find the characteristic equation, find the roots, write the general form, then solve a system of linear equations to find the particular solution.