Random Variables

1. $V(X+Y) = V(X) + V(Y)$ for independent random variables X, Y.
2. $V(aX) = a^2 V(X)$ for any $a \in \mathbb{R}$ and random variable X.
3. $V(X+a) = V(X)$ for any $a \in \mathbb{R}$ and random variable X.

Exercises

1. Prove the formulas $4, 5, 6$ above.

   4 Use the formula $V(X) = E(X^2) - E(X)^2$ and independence ($E(XY) = E(X)E(Y)$):

$$\begin{aligned} V(X+Y) &= E((X+Y)^2) - (E(X+Y))^2 \\ &= E(X^2 + 2XY + Y^2) - (E(X) + E(Y))^2 \\ &= E(X^2) + 2E(XY) + E(Y^2) - (E(X)^2 + 2E(X)E(Y) + E(Y)^2) \\ &= (E(X^2) - E(X)^2) + (E(Y^2) - E(Y)^2) + 2(E(XY) - E(X)E(Y)) \\ &= V(X) + V(Y) \end{aligned}$$

   5 Use the definition $V(X) = E(X^2) - E(X)^2$:

$$\begin{aligned} V(aX) &= E((aX)^2) - E(aX)^2 \\ &= E(a^2X^2) - (aE(X))^2 \\ &= a^2E(X^2) - a^2E(X)^2 \\ &= a^2(E(X^2) - E(X)^2) = a^2V(X) \end{aligned}$$

   6 Use the definition again (it's mostly the same as 4):

$$\begin{aligned} V(a+X) &= E((a+X)^2) - (E(a+X))^2 \\ &= E(a^2 + 2aX + X^2) - (E(a) + E(X))^2 \\ &= a^2 + 2aE(X) + E(X^2) - (E(a)^2 + 2E(a)E(X) + E(X)^2) \\ &= a^2 + 2aE(X) + E(X^2) - (a^2 + 2aE(X) + E(X)^2) \\ &= E(X^2) - E(X)^2 = V(X) \end{aligned}$$

2. Consider the experiment of flipping a fair coin ($\frac{1}{2}$ probability heads and tails). Then the sample space of a single experiment is $S = \{\text{heads, tails}\}$ and our probability distribution assigns $\frac{1}{2}$ to each.

   Consider the following random variables on S:

   (a) X defined by $X(\text{tails}) = 0$ and $X(\text{heads}) = 1$.

   (b) Y defined by $Y(\text{tails}) = -1$ and $Y(\text{heads}) = 1$.

   (a) Compute the new random variables XY and $X + Y$ on S.
   $XY(\text{tails}) = X(\text{tails})Y(\text{tails}) = 0 * -1 = 0$
   $XY(\text{heads}) = X(\text{heads})Y(\text{heads}) = 1 * 1 = 1$
   $(X + Y)(\text{tails}) = X(\text{tails}) + Y(\text{tails}) = 0 + -1 = -1$
   $(X + Y)(\text{heads}) = X(\text{heads}) + Y(\text{heads}) = 1 + 1 = 2$

   (b) Now compute $E(XY)$ and $E(X)E(Y)$.
   $E(XY) = p(\text{tails})XY(\text{tails}) + p(\text{heads})XY(\text{heads}) = \frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2}$
   $E(X+Y) = p(\text{tails})(X+Y)(\text{tails}) + p(\text{heads})(X+Y)(\text{heads}) = \frac{1}{2}(-1) + \frac{1}{2}(2) = \frac{1}{2}$
   (We could also just have used $E(X + Y) = E(X) + E(Y)$ instead.)

   (c) Now compute $V(X + Y)$ and $V(X) + V(Y)$.
   $V(XY) = \frac{1}{2}(0 - \frac{1}{2})^2 + \frac{1}{2}(1 - \frac{1}{2})^2 = \frac{1}{4}$
   $V(X + Y) = \frac{1}{2}(-1 - \frac{1}{2})^2 + \frac{1}{2}(2 - \frac{1}{2})^2 = \frac{9}{4}$

   (d) Are X and Y independent?
   X and Y are not independent. For example: $P(X = 0, Y = 1) = 0$ because X is never 0 when Y is 1. However, $P(X = 0)P(Y = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq 0$.

3. Consider the experiment of rolling a 6-sided die until you roll a 3. What is the sample space S? What is the probability distribution on S? How many times do you expect to roll the die on average?

   This is a (slight) modification to the usual geometric distribution, because we are considering sequences such as 1263 and 6453 as different results whereas in the geometric distribution we would consider these the same because the number of failures is the same.

   The sample space for this situation is the set of all strings on the alphabet $\{1, 2, 3, 4, 5, 6\}$ that end in 3. (e.g. it looks like $\{3, 13, 23, 43, 53, 63, 113, 123, \dots\}$.)

   The probability distribution on S is that a string of length n has probability $\frac{1}{6^n}$.

   The computation for the expected number of rolls is the same as for the geometric distribution (don't worry about how to compute this...it's not particularly difficult, but it's probably not relevant for this class). The value for the geometric distribution (which you calculated in class) is $6 = \frac{1}{\frac{1}{6}}$ since $\frac{1}{6}$ is the chance of success.