Probability

Before, we defined probability as the number of 'good' outcomes (size of an event) divided by the total number of outcomes. We do this more formally as: let S be a (countable) sample space. A *probability distribution* is a function $p : S \to \mathbb{R}$ such that:

1. $0 \leq p(s) \leq 1$ for all $s \in S$ (all probabilities should be nonnegative and less than 1)
2. $\sum_{s \in S} p(s) = 1$ (the sum of all probabilities should be 1)

We call $p(s)$ the probability of an outcome $s \in S$. For an event $E \subset S$, we define the probability as $p(E) = \sum_{s \in E} p(s)$.

Exercises

For the following, let S be a sample space, and p a probability distribution on S.

1. For any event $E \subset S$, show that $p(\overline{E}) = 1 - p(E)$.

2. For any events $E_1, E_2 \subset S$, show that $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$.

3. For any parwise disjoint events $E_1, E_2, \cdots \subset S$, show that $p(\cup_i E_i) = \sum_i p(E_i)$.

4. We say that events $E_1, E_2 \subset S$ are *independent* if $p(E_1 \cap E_2) = p(E_1)p(E_2)$. Under what situation can two disjoint events $E_1, E_2 \subset S$ be independent?

Conditional Probability

Let $E, F \subset S$ be events. We often want to know the probability of E given that F occurs. This is the *conditional probability of E given F*, and we write it as $P(E \mid F)$. (We usually assume $p(F) > 0$ in this situation.) To compute this, we use $p(E \mid F) = \frac{p(E \cap F)}{p(F)}$.

Exercises

1. Prove that if $p(E \mid F) = p(E)$, then E and F are independent events.

2. A *Bernoulli trial* is an experiment with two outcomes, one ("success") with fixed probability p and the other ("failure") with probability $1-p$. Prove that the probability of k successes in n independent Bernoulli trials is $\binom{n}{k} p^k (1-p)^{n-k}$. What is the sum $\sum_{i=0}^{n} \binom{n}{k} (p^k)(1-p)^{n-k}$ in terms of a simple expression?

Bayes' Theorem

Bayes' Theorem is a statement relating the conditional probabilities: $p(E \mid F)$ and $p(F \mid E)$ for events $E, F \subset S$. In class, you saw the statement: $p(E \mid F) = \frac{p(F|E)p(E)}{p(F|E)p(E) + p(F|\overline{E})p(\overline{E})}$.

Exercises

1. Show that $p(F \mid E)p(E) + p(F \mid \overline{E})p(\overline{E}) = p(F)$. Hence we can also write Bayes' Theorem as $p(E \mid F) = \frac{p(F|E)p(E)}{p(F)}$. Prove this form of Bayes' Theorem using the definition of conditional probability.

   (Often it will be more convenient to use one form over another for computations. For example, some cases may give $p(F)$ directly whereas other cases may cause computing the larger expression $p(F \mid E)p(E) + p(F \mid \overline{E})p(\overline{E})$ to be more intuitive.)