Probability

Exercises

1. You have 10 cards labelled $1-10$ and placed face-down. The game can be played in one of two ways:

   (a) You select 5 out of the 10 cards all at once.

   (b) You select 5 cards one at a time.

   If you win by selecting the cards $1, 2, 3, 4, 5$ in any order, what is the probability of winning in each scenario? What is the sample space in each scenario?

   For (a), the sample space is the set of 5 element subsets of $\{1, 2, \ldots, 10\}$. The size of the sample space is $\binom{10}{5}$. There is only 1 way to get a 5 element subset of $\{1, 2, \ldots, 10\}$ containing $\{1, 2, 3, 4, 5\}$, so the chance of winning is $\frac{1}{\binom{10}{5}}$.

   For (b), the sample space is 5 element permutations of $\{1, 2, \ldots, 10\}$. The size of the sample space is $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$. There are 5! ways to select the cards $1, 2, 3, 4, 5$ because you can draw them in any order, so the chance of winning is $\frac{5!}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}$.

2. In the same setup as above, you now have to choose the cards $1, 2, 3, 4, 5$ either in decreasing or asccending order to win. This time, you are only allowed to select cards one at a time. What is the chance of winning? What is the sample space?

   The sample space is the same as (b) in the previous problem. There are only 2 ways to get $1, 2, 3, 4, 5$ in ascending/decreasing order, so the chance of winning is $\frac{2}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}$.

3. You have a deck of 52 cards labelled $1-52$. If you pick 5 cards all at once, what is the chance that 4 of the cards you drew are congruent mod 13? What is the sample space? If you instead draw cards one at a time, what is the probability that the first four are congruent mod 13?

   The sample space is the set of 5 element subsets of $\{1, 2, \ldots, 52\}$. Picking all 5 at once, the chance that 4 are congruent mod 13 is the same as picking a number mod 13 (so $0, 1, \ldots, 12$) because this forces the choice of the four numbers. Then we still need to choose one of the 48 remaining cards for the fifth. Hence the chance of winning is $\frac{13*48}{\binom{52}{5}}$.

   If you draw one at a time, and only the first four count, then the chance of winning is $\frac{52*3*2}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$ (first you can pick any card, then the rest are determined).

Generalized Monty Hall

In class, you studied the following question: On a game show, there are three doors. One leads to a prize, while the others lead to nothing. The game goes as follows: you first pick one door. The host then reveals a door leading to nothing. Finally, you can switch doors (there's one remaining) or stay with your choice. Perhaps counter-intuitively, switching doubles your chances of winning!

Think about the following generalization: Suppose you have n doors, one leading to a prize and the other $n-1$ leading to nothing. As before, you first choose a door. The host then reveals to you k doors (where $1 \leq k \leq n-2$). What are the probabilities of winning if you switch or if you stay?

1. Try to visualize the situation when n is large, and $k = n-2$. (For example, if $n = 1,000,000$, you can be almost certain that your first door was incorrect. The host then reveals $999,998$ doors leading to nothing, leaving one mysterious door. Were they avoiding that door on purpose, or were they forced to?) What is the chance of winning in this case, both for switching and staying?

   For staying, the chance of winning is $\frac{1}{1{,}000{,}000}$.

   For switching, the chance of winning is $\frac{999{,}999}{1{,}000{,}000}$.

2. Now consider the general case: any positive integer $n \geq 3$ and any k between 1 and $n-2$. What are the chances of winning of staying and switching?

   For staying, the chance of winning is still $\frac{1}{n}$.

   For switching, the chance of winning is $\frac{n-1}{n}\frac{1}{n-k-1}$. (Note that this is strictly better than $\frac{1}{n}$ in the given circumstances.)

3. Now suppose the host gives you a choice: you can either pick first and then the k doors will be revealed to you, or you can have the k doors revealed before you pick. What is the chance of winning in each scenario?

   You should pick first and then have the option to stay or switch.

   If you do not pick first, then the chance of winning is $\frac{1}{n-k}$ whereas the chance of winning if you pick first is $\frac{n-1}{n}\frac{1}{n-k-1}$.

   To compare them, note that $\frac{n-1}{n} > \frac{n-k-1}{n-k}$.