

Probability

An *experiment* is a procedure that produces one of a given set of outcomes. The *sample space* of an experiment is the set of all possible outcomes. An *event* is a subset of the sample space. If the sample space, S , is finite, and each outcome in the sample space is equally likely, then the probability of an event $E \subset S$ is $P(E) = \frac{|E|}{|S|}$. (Although it makes sense to think about infinite sample spaces of non-equally likely outcomes, we will study them later.)

Exercises

1. You have 10 cards labelled 1 – 10 and placed face-down. The game can be played in one of two ways:
 - (a) You select 5 out of the 10 cards all at once.
 - (b) You select 5 cards one at a time.

If you win by selecting the cards 1, 2, 3, 4, 5 in any order, what is the probability of winning in each scenario? What is the sample space in each scenario?

2. In the same setup as above, you now have to choose the cards 1, 2, 3, 4, 5 either in decreasing or ascending order to win. This time, you are only allowed to select cards one at a time. What is the chance of winning? What is the sample space?
3. You have a deck of 52 cards labelled 1 – 52. If you pick 5 cards all at once, what is the chance that 4 of the cards you drew are congruent mod 13? What is the sample space? If you instead draw cards one at a time, what is the probability that the first four are congruent mod 13?

Generalized Monty Hall

In class, you studied the following question: On a game show, there are three doors. One leads to a prize, while the others lead to nothing. The game goes as follows: you first pick one door. The host then reveals a door leading to nothing. Finally, you can switch doors (there's one remaining) or stay with your choice. Perhaps counter-intuitively, switching doubles your chances of winning!

Think about the following generalization: Suppose you have n doors, one leading to a prize and the other $n - 1$ leading to nothing. As before, you first choose a door. The host then reveals to you k doors (where $1 \leq k \leq n - 2$). What are the probabilities of winning if you switch or if you stay?

1. Try to visualize the situation when n is large, and $k = n - 2$. (For example, if $n = 1,000,000$, you can be almost certain that your first door was incorrect. The host then reveals 999,998 doors leading to nothing, leaving one mysterious door. Were they avoiding that door on purpose, or were they forced to?) What is the chance of winning in this case, both for switching and staying?
2. Now consider the general case: any positive integer $n \geq 3$ and any k between 1 and $n - 2$. What are the chances of winning of staying and switching?
3. Now suppose the host gives you a choice: you can either pick first and then the k doors will be revealed to you, or you can have the k doors revealed before you pick. What is the chance of winning in each scenario?