Counting

Exercises

For these exercises, let $S = \{1, \dots, 10\}$ be the set of integers from 1 to 10. Explain first which of the above rules apply, and then use them to count the number of subsets.

1. How many subsets of S are there?

   Product rule. 2^{10}.

2. How many subsets of S are there containing 1? Containing 10?

   Product rule. 2^9 in both cases.

3. How many subsets of S contain both 1 and 10?

   Product rule. 2^8.

4. How many subsets of S contain 1 or 10?

   Inclusion Exclusion. $2^9 + 2^9 - 2^8 = 2^{10} - 2^8$.

5. How many subsets of S contain neither 1 nor 10?

   Product rule. 2^8.

   (Alternatively, it's the complement of the set in number 4, so it's $2^{10} - (2^{10} - 2^8) = 2^8$.)

Exercises

1. What is the smallest integer n such that any subset of $\{1, 2, \dots, 9\}$ with n elements is guaranteed to have two numbers adding to 10? (Note: this is reworded from the worksheet given out in class.)

   In the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, there are 4 pairs of distinct numbers adding up to 10: $(1, 9); (2, 8); (3, 7); (4, 6)$. Then for any subset of size at most 5, we are not guaranteed to have a pair adding up to 10, because we can pick one number from each pair.

   However, for any subset of size 6, we are forced to pick at least two elements from one of the listed pairs, so there will be a pair adding to 10.

2. Let $a_1, \dots, a_n \in \mathbb{Z}$. Show that there are $1 \le b \le c \le n$ where $a_b + a_{b+1} + \cdots + a_c$ is divisible by n. (Hint: consider the sums $s_i = a_1 + \cdots + a_i$ for $1 \le i \le n$. What happens if none of them are divisible by n?)

   Consider the following sums:

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= s_1 + a_2 + \cdots + a_n
\end{aligned}
$$

There are n sums here. Now let's think about the possible remainders $\mod n$. We'll consider two cases.

Case 1: $n \mid s_i$ for some i. Since each s_i was a consecutive sum $a_1 + \cdots + a_i$, then we are done because we have found a consecutive sum divisible by n.

Case 2: $n \nmid s_i$ for any i. Then by definition, $s_i \mod n \neq 0$ for all i. The possible values for $a \mod n$ are $0, 1, \ldots, n-1$ for any integer $a \in \mathbb{Z}$.

Therefore the possible remainders $s_i \mod n$ are in the set $\{1, \ldots, n-1\}$, because we have already ruled out the case where $s_i \mod n = 0$.

Now there are n sums and $n-1$ possible remainders, so by the pigeonhole principle, there are at least two distinct sums with the same remainder. Call them s_i and s_j where $i \neq j$ and WLOG $i < j$. (Otherwise if $i > j$, then switch i and j.)

This means that $s_i \mod n = s_j \mod n \implies s_i \equiv s_j \mod n \implies n \mid s_j - s_i$. However, we know that $s_j - s_i = (a_1 + \cdots + a_j) - (a_1 + \cdots + a_i) = a_{i+1} + \cdots + a_j$. Therefore, n divides this consecutive sum, and we are done.

Thus in all cases, we have some consecutive sum divisible by n, so we are done.

3. Suppose you have 3 spheres, and 7 cubes, each labelled with a number between 0 and 9. (The worksheet given out in class had a typo here. It used to be 19.) Use the pigeonhole principle to show that there are at least two different sphere-cube pairs whose sums are equal. Is this still true with 6 cubes instead of 7?

   There are 21 distinct sphere-cube pairs. There are 19 possible sums: $0, 1, \ldots, 18$. Since each pair has a sum, then we have 21 pairs which map to 19 sums, so by the pigeonhole principle there are at least two pairs that have the same sum.

   This is still true with 6 cubes, but we can no longer just apply the pigeonhole principle, and the problem becomes much harder. Don't worry about how to prove this.

4. Suppose there is a hotel with infinitely many rooms, each room labelled with a positive, even integer. If there are also infinitely many people, each one labelled with a positive integer, who must get a room, does the pigeonhole principle say that some room must have at least two people in it?

   No. There's a bijection between the positive, even integers and the positive integers (mulitply by 2), so each person can get their own room, or we can even have empty rooms! The problem here is that we do not know the pigeonhole principle for infinitely many objects/boxes, so we cannot apply it to this situation.