Counting

1. Product Rule: A sequence of two tasks, one of which can be performed in n_1 ways and the other of which can be performed in n_2 ways, can be performed in a total of $n_1 n_2$ ways.
2. Sum Rule: A single task, which can be performed in n_1 ways or in n_2 other ways, can be performed in a total of $n_1 + n_2$ ways.
3. Inclusion-Exclusion: A single task, which can be performed in n_1 ways or n_2 ways (not necessarily distinct), can be performed in a total of $n_1 + n_2$ ways minus the number of ways counted twice.

Exercises

For these exercises, let $S = \{1, \ldots, 10\}$ be the set of integers from 1 to 10. Explain first which of the above rules apply, and then use them to count the number of subsets.

1. How many subsets of S are there?
2. How many subsets of S are there containing 1? Containing 10?
3. How many subsets of S contain both 1 and 10?
4. How many subsets of S contain 1 or 10?
5. How many subsets of S contain neither 1 nor 10?

1. Pigeonhole principle: Putting $n+1$ objects into n boxes always results in at least one bin with at least 2 objects in it.
2. Generalized: Placing n objects into k boxes results in at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Exercises

1. What is the smallest integer n such that any subset of $\{1, 2, \ldots, 9\}$ with n elements is guaranteed to have two numbers adding to 10? (Note: this is reworded from the worksheet given out in class.)

2. Let $a_1, \ldots, a_n \in \mathbb{Z}$. Show that there are $1 \leq b \leq c \leq n$ where $a_b + a_{b+1} + \cdots + a_c$ is divisible by n. (Hint: consider the sums $s_i = a_1 + \cdots + a_i$ for $1 \leq i \leq n$. What happens if none of them are divisible by n?)

3. Suppose you have 3 spheres, and 7 cubes, each labelled with a number between 0 and 9. (The worksheet given out in class had a typo here. It used to be 19.) Use the pigeonhole principle to show that there are at least two different sphere-cube pairs whose sums are equal. Is this still true with 6 cubes instead of 7?

4. Suppose there is a hotel with infinitely rooms, each room labelled with a positive, even integer. If you have infinitely many people, each one labelled with a positive integer, does the pigeonhole principle say that some room must have at least two people in it?