More Induction

1. Assume you only know that $\frac{d}{dx}(x^0) = 0$ and also that the product rule is true. Is it possible to use induction to prove $\frac{d}{dx}(x^n) = nx^{n-1}$ for all integers $n \geq 0$?

   No. At the inductive step for $n = 0$ (i.e. proving $P(1)$ assuming $P(0)$), we need to use $P(1)$ to complete the proof, because we'd like to reduce $\frac{d}{dx}(x^1) = x^a \frac{d}{dx} x^b + x^b \frac{d}{dx} x^a$ for some $a + b = 1$, but the only such integers are 0 and 1.

2. Now assume you only know that $\frac{d}{dx}(x) = 1$ and also that the product rule is true. Now use induction to prove $\frac{d}{dx}(x^n) = nx^{n-1}$ for all integers $n \geq 1$.

   Yes. Here is the proof:

   Base Case: $n = 1$, then $P(1) : \frac{d}{dx}(x^1) = 1x^0 = 1$ which is given to us.

   Inductive Hypothesis: Assume $P(n)$.

   Inductive Step: Show $P(n+1)$. Since $n \geq 1$, then $n + 1 \geq 2$. Therefore:

   $$\frac{d}{dx}(x^{n+1}) = \frac{d}{dx}(x^n x) = x^n \frac{d}{dx}(x) + x \frac{d}{dx}(x^n) = x^n + nx * x^{n-1}$$

   where the last equality follows from applying $P(1)$ and $P(n)$. This is okay since $P(1)$ is the base case, and $P(n)$ is our inductive hypothesis. Then:

   $$x^n + nx * x^{n-1} = x^n + nx^n = x^n(n+1) = (n+1)x^n$$

3. Is it possible to extend your proof above to all real numbers? No. The real numbers with the usual order is not a well-ordered set, so we cannot apply induction.

4. Show that $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$ for all integers $n \geq 0$.

   Base Case: $n = 0$, then $P(0) : 0 = (0+1)! - 1 = 1 - 1 = 0$.

   Inductive Hypothesis: Assume $P(n)$.

   Inductive Step: Show $P(n+1)$.

   $$\begin{aligned} 1(1!) + 2(2!) + \cdots + (n+1)(n+1)! &\overset{P(n)}{=} ((n+1)! - 1) + (n+1)(n+1)! \\ &= (n+1)!(1 + n + 1) - 1 = (n+2)(n+1)! - 1 \\ &= (n+2)! - 1 \end{aligned}$$

5. Show that $9 \mid 4^n + 15n - 1$ for all integers $n \geq 0$.

   Base case: $n = 0$, $P(0)$: does 9 divide $4^0 + 15(0) - 1$? $4^0 + 15(0) - 1 = 0$, and $9 \mid 0$.

   Inductive Hypothesis: Assume $P(n)$.

   Inductive Step: Show $P(n+1)$.

   $$\begin{aligned} 4^{n+1} + 15(n+1) - 1 &= 4 * 4^n + 15n + 15 - 1 = (3+1)4^n + 15n + 15 - 1 \\ &= 3 * 4^n + 15 + (4^n + 15n - 1) \end{aligned}$$

   Since $9 \mid 4^n + 15n - 1$ by $P(n)$, then it is enough to show $9 \mid 3 * 4^n + 15 = 3(4^n + 5)$. Since $4^n + 5 \equiv 1^n + 2 \equiv 0 \mod 3$, then $9 \mid 3 * 4^n + 15$.

Recursive Definitions

1. Recall that an arithmetic progression is of the form $a_n = a + nd$, where $a, d \in \mathbb{R}$ and $n \in \mathbb{N}$. Write this sequence using a recursive definition.

   $a_{n+1} = a_n + d$, and $a_0 = a$.

2. Recall that a geometric progression is of the form $a_n = ar^n$, where $a, r \in \mathbb{R}$ and $n \in \mathbb{N}$. Write this sequence using a recursive definition.

   $a_{n+1} = ra_n$, and $a_0 = a$.

3. Give a recursive definition of the set $\mathbb{N}$.

   $\mathbb{N}$ is the set S where (base case) $0 \in S$ and (inductive step) $x \in S \implies x + 1 \in S$.

4. Recall the Fibonacci sequence defined by $f_0 = 0, f_1 = 1$, and $f_n + f_{n+1} = f_{n+2}$ for all $n \in \mathbb{N}$. Prove (using induction) that $f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$ for $n \in \mathbb{Z}^+$.

   We'll prove it by induction.

   Base Case: $n = 1$, $P(1) : f_0 f_1 + f_1 f_2 = 0 * 1 + 1 * 1 = 1^2 = f_2^2$.

   Inductive Hypothesis: Assume $P(n)$.

   Inductive Step: Show $P(n+1)$.

$$
\begin{aligned}
&f_0 f_1 + f_1 f_2 + \cdots + f_{2n-1} f_{2n} + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+1} \\
&\qquad \overset{P(n)}{=} f_{2n}^2 + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2} = f_{2n}(f_{2n} + f_{2n+1}) + f_{2n+1} f_{2n+2} \\
&\qquad = f_{2n}(f_{2n+2}) + f_{2n+1} f_{2n+2} = (f_{2n} + f_{2n+1}) f_{2n+2} = f_{2n+2}^2
\end{aligned}
$$

5. Prove (using induction) that $f_{n-1} f_{n+1} - f_n^2 = (-1)^n$ for all $n \in \mathbb{Z}^+$.

   We'll prove it by induction.

   Base Case: $n = 1$, $P(1) : f_0 f_2 - f_1^2 = 0 * 1 - 1 * 1 = (-1)^1$.

   Inductive Hypothesis: Assume $P(n)$.

   Inductive Step: Show $P(n+1)$.

$$
\begin{aligned}
f_n f_{n+2} - f_{n+1}^2 &= f_n(f_{n+1} + f_n) - f_{n+1}(f_n + f_{n-1}) \\
&= f_n f_{n+1} + f_n^2 - f_{n+1} f_n - f_{n+1} f_{n-1} \\
&= (f_n f_{n+1} - f_{n+1} f_n) + (f_n^2 - f_{n+1} f_{n-1}) \\
&= (f_n f_{n+1} - f_n f_{n+1}) - (f_{n-1} f_{n+1} - f_n^2) \\
&= 0 - (f_{n-1} f_{n+1} - f_n^2) = -(f_{n-1} f_{n+1} - f_n^2) \\
&\overset{P(n)}{=} = -(-1)^n = (-1)(-1)^n = (-1)^{n+1}
\end{aligned}
$$