Functions (cont.)

Recall: a function $f : A \to B$ is injective (or one-to-one) if $f(a_1) = f(a_2) \implies a_1 = a_2$ for any $a_1, a_2 \in A$. A function $f : A \to B$ is surjective (or onto) if for every $b \in B$, there is $a \in A$ such that $f(a) = b$.

Definitions

1. A function is *bijective* if it is both surjective and injective. We call such functions bijections or one-to-one correspondences.
2. Let $f : A \to B$ be a function. The *inverse function*, if it exists, is the function $f^{-1} : B \to A$ defined by $f^{-1}(b) = a$ if $f(a) = b$.
3. Let $f : A \to B$ and $g : B \to C$ be functions. The *composition* is the function $g \circ f : A \to C$ defined by $g \circ f(a) = g(f(a))$.
4. A sequence is a function from (a subset of) $\mathbb{Z}$ to a set S.

Exercises

1. Let $f : A \to B$ be a function. What condition on f do you need for the inverse function to exist? If the inverse exists, what are the compositions $f \circ f^{-1}$ and $f^{-1} \circ f$? (i.e. what are the domains and codomains, and do you know another name for these functions?)

   f must be bijective in order for the inverse to exist. (If f is not injective, then $f^{-1}(b)$ is not uniquely defined, so $f^{-1}(b)$ will not be a function. If f is not surjective, then there is some $b \in B$ where $f^{-1}(b)$ will have no possible output.)

   $f \circ f^{-1} : B \to B$ and $f^{-1} \circ f : A \to A$ are both the identity functions on their respective sets.

2. Let A be a set. Consider the set $S = \{f : A \to \{0, 1\}\}$ of functions from A to $\{0, 1\}$. Can you identify S in terms of a set construction you already know?

   S is just the power set of A. Think of a function $f : A \to \{0, 1\}$ as saying which elements of A are in that subset (1 if it is in, 0 if not).

3. Decide whether the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = \lfloor \frac{x}{3} \rfloor$ is surjective. (If yes, give a proof.) Is it injective?

   This function is surjective. For any $n \in \mathbb{Z}$, $f(3n) = \lfloor \frac{3n}{3} \rfloor = \lfloor n \rfloor = n$. This function is not injective. For example, $f(0) = f(1) = 0$.

Cardinality

Definitions

1. Two sets A and B have the same cardinality if there is a bijection $f : A \to B$, and we say that A and B are in bijection.

2. A set is *countable* if it is in bijection with a subset of the $\mathbb{Z}^+$. Otherwise we call it uncountable. (There are many kinds of uncountable, but we will not worry about them.) Common countable sets are $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$. Common uncountable sets are $\mathbb{R}, \mathbb{C}$.

Exercises

1. Show that any subset of a countable set is countable.

   Let A be our countable set, and $B \subseteq A$ our subset. Since A is countable, we have a bijection $f : A \to S$ between A and some subset of the positive integers $S \subseteq \mathbb{Z}^+$.

   Define the *restriction* of $f : A \to S$ to B (so we're restricting the domain of f to B) as the function $f|_B : B \to S$ defined by $f|_B(b) = f(b)$ for $b \in B$. Then B is in bijection with the image of $f|_B$, which is a subset of a subset of $\mathbb{Z}^+$, so is a subset of $\mathbb{Z}^+$. Hence B is again countable.

2. Let A be any set. Show that there is no surjection $f : A \to \mathcal{P}(A)$.

   Proof: We're going to use proof by contradiction. The style is similar to that of Russell's paradox.

   Suppose for contradiction that there is some surjection $f : A \to \mathcal{P}(A)$. Then (this is the tricky step!) consider the set $B = \{x \in A \mid x \notin f(x)\}$.

   Notice that $B \subseteq A$, so $B \in \mathcal{P}(A)$, and because f is surjective there is some $a \in A$ such that $f(a) = B$. Now we have two cases: either $a \in B$ or $a \notin B$. Let's look what happens:

   (a) If $a \in B$: Then by definition of B, $a \notin f(a)$. However, $f(a) = B$, so $a \notin B$.
   (b) If $a \notin B$: Then since $f(a) = B$, we have $a \notin f(a)$. But then $a \in B$.

   Let's phrase this in English. If a is an element of B, then it must not be an element of B. On the other hand, if a is not an element of B, then it must be an element of B. These are both impossible, so we have reached a contradiction. Hence our assumption that there was some surjection $f : A \to \mathcal{P}(A)$ is false.

3. Use the above to show that the power set of the positive integers is uncountable.

   Let $A = \mathbb{Z}^+$ in the problem above. Then the result of the previous problem says that there is no surjection $f : \mathbb{Z}^+ \to \mathcal{P}(\mathbb{Z}^+)$. But any bijection is also a surjection, so there can be no bijection. Hence $\mathcal{P}(\mathbb{Z}^+)$ is not countable, so it is uncountable.