Sets

A set is an unordered collection of objects. We write $x \in S$ when x is an element of a set S. Some common sets are $\varnothing, \mathbb{N}, \mathbb{Z}, \mathbb{Q}\, \mathbb{R}, \mathbb{C}$. Let A and B be sets. Recall the following:

Definitions

1. $A \subseteq B$ if and only if every $x \in A$ is also in B
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
3. $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$ is the union
4. $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$ is the intersection
5. $|A|$ is the cardinality of A. It is an integer if A is finite, and infinite otherwise. (We will learn how to distinguish infinite sets later.)
6. $\mathcal{P}(S)$ is the power set of S, the set of all subsets of S
7. $A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$ is the Cartesian Product.
8. $\overline{A}$ is the complement of A, the set of all elements not in A. Remember that this depends on the universal set!

Exercises

1. Let A and B be finite sets. (Recall: this means $|A|$ and $|B|$ are integers.) What are the cardinalities of the following sets? If the answer is "it depends," then give a bound.
   (a) $A \cup B$
   $|A \cup B| = |A| + |B| - |A \cap B|$
   (b) $A \cap B$
   Use above: $|A \cap B| = |A| + |B| - |A \cup B|$
   (c) $\mathcal{P}(A)$
   $|\mathcal{P}(A)| = 2^{|A|}$ (Think of a subset of A as a choice for each element of A whether that element is or is not in the subset.)
   (d) $A \times B$
   $|A \times B| = |A||B|$
2. Express $A - B$ using only intersections, unions, and complements.
   $A - B = A \cap \overline{B}$
3. Prove De Morgan's laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
   I'll do the first one only (the second one is similar). $x \in \overline{A \cup B} \iff x \notin A \cup B \iff x \notin A$ and $x \notin B \iff x \in \overline{A}$ and $x \in \overline{B} \iff x \in \overline{A} \cap \overline{B}$.

4. Prove the distributive laws for sets: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (union over intersection) and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (intersection over union). Can you prove these using the distributive laws from logic?

   Again I'll do only the first one (the second one is very similar). $x \in A \cup (B \cap C) \implies x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. On the other hand, if $x \in B \cap C$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$.

   Now let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. Then ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$).

   Consider two cases: $x \in A$ and $x \notin A$. If $x \in A$, then $x \in A \cup (B \cap C)$. If $x \notin A$, then we must have $x \in B$ and $x \in C$. Then $x \in C \cap B$, so $x \in A \cup (B \cap C)$.

5. Let A_i be the set of all integers greater than i. Describe the sets $\cup_{i=0}^{n} A_i$, $\cap_{i=0}^{n} A_i$, $\cup_{i=0}^{\infty} A_i$, and $\cap_{i=0}^{\infty} A_i$. (For the last two, the notation $\cup_{i=0}^{\infty}$ and $\cap_{i=0}^{\infty}$ means for every integer greater than or equal to 0. We won't be too worried with infinite unions and intersections in this class.)

   $\cup_{i=0}^{n} A_i = A_0$.

   $\cap_{i=0}^{n} A_i = A_n$

   $\cup_{i=0}^{\infty} A_i = A_0$

   $\cap_{i=0}^{\infty} A_i = \varnothing$

Functions

Fix sets A and B. A function from A to B is an assignment of a unique element of B to each element of A, and we write this as $f : A \to B$ for the function and $f(a) = b$ for an evaluation of the function at an element $a \in A$.

Definitions

1. A is called the domain of f
2. B is called the codomain of f
3. If $X \subseteq A$, then $f(X) \subseteq B$ is called the image of X
4. The range of f is the image of A, $f(A) \subseteq B$.
5. The preimage of $Y \subseteq B$ is $\{a \in A \mid f(a) \in Y\}$.
6. f is injective (or one-to-one) if $f(a_1) = f(a_2) \implies a_1 = a_2$ for any $a_1, a_2 \in A$.
7. f is surjective (or onto) if for every $b \in B$, there is some $a \in A$ with $f(a) = b$.

Exercises

1. If $f : A \to B$ is surjective, what is the relationship between the range and the codomain?

   The range is equal to the codomain.

2. For a function $f : A \to B$, what kind of object is the preimage of an element $b \in B$?

   The preimage is a set.

3. Let $A_1, A_2 \subseteq A$ for a function $f : A \to B$. Is $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$? What about $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$?

   Yes to the first. $x \in f(A_1 \cup A_2) \iff \exists a \in A_1 \cup A_2$ such that $f(a) = x \iff \exists a \in A_1$ or $\exists a \in A_2$ such that $f(a) = x \iff x \in f(A_1)$ or $x \in f(A_2) \iff x \in f(A_1) \cup f(A_2)$.

   No to the second. (There are many other counterexamples – this is not the only one!) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 0$. Let $A_1 = \{1, 2, 3\}$ and $A_2 = \{4, 5, 6\}$. Then $A_1 \cap A_2 = \varnothing$, and $f(A_1 \cap A_2) = f(\varnothing) = \varnothing$. However, $f(A_1) = \{0\}$ and $f(A_2) = \{0\}$, so $f(A_1) \cap f(A_2) = \{0\} \neq \varnothing$.

4. Describe how $f : A \to B$ can be thought of as a subset of $A \times B$.

   We can think of $f : A \to B$ as a set of pairs (a, b) where $f(a) = b$. This is also known as the graph of a function.

5. Let $B_1, B_2 \subseteq B$ for a function $f : A \to B$. Is the preimage of $B_1 \cup B_2$ equal to the union of the preimages of B_1 and B_2? If the preimage of $B_1 \cap B_2$ equal to the intersection of the preimages of B_1 and B_2?

   Yes to the first. x is in the preimage of $B_1 \cup B_2 \iff f(x) \in B_1 \cup B_2 \iff f(x) \in B_1$ or $f(x) \in B_2 \iff x$ is in the preimage of B_1 or x is in the preimage of $B_2 \iff x$ is in the union of the preimages of B_1 and B_2.

   Yes to the second. x is in the preimage of $B_1 \cap B_2 \iff f(x) \in B_1 \cap B_2 \iff f(x) \in B_1$ and $f(x) \in B_2 \iff x$ is in the preimage of B_1 and x is in the preimage of $B_2 \iff x$ is in the intersection of the preimages of B_1 and B_2.

6. Decide whether the following are functions. If they are, classify them as surjective, injective, both, or neither:

   (a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x + 1$
   This is a function. It is both surjective and injective (we call these functions bijective).

   (b) $f : \mathbb{Z} \to \mathbb{Z}_{>0}$ defined by $f(x) = 2|x|$
   This is not a function. $f(0) = 2|0| = 0$ is not in the codomain.

   (c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$
   This is a function. It is both surjective and injective.

   (d) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = \frac{1}{x}$.
   This is not a function. $f(0) = \frac{1}{0}$ is not a number, so it is not in the codomain.

(e) $f : \mathbb{R} \to \mathbb{R}_{>0}$ defined by $x \to e^x$ ($\mathbb{R}_{>0}$ is the set of positive real numbers)
This is a function. It is both surjective and injective.

(f) $f : \mathbb{R} \to \mathbb{R}$ defined by $x \to \log(x)$
This is not a function. For us, the logarithm is not defined for negative numbers. (Alternatively, if you do define it for negative numbers, it is not a real-valued function.)