Sets

A set is an unordered collection of objects. We write $x \in S$ when x is an element of a set S. Some common sets are $\varnothing, \mathbb{N}, \mathbb{Z}, \mathbb{Q}\, \mathbb{R}, \mathbb{C}$. Let A and B be sets. Recall the following:

Definitions

1. $A \subseteq B$ if and only if every $x \in A$ is also in B
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
3. $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$ is the union
4. $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$ is the intersection
5. $|A|$ is the cardinality of A. It is an integer if A is finite, and infinite otherwise. (We will learn how to distinguish infinite sets later.)
6. $\mathcal{P}(S)$ is the power set of S, the set of all subsets of S
7. $A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$ is the Cartesian Product.
8. $\overline{A}$ is the complement of A, the set of all elements not in A. Remember that this depends on the universal set!

Exercises

1. Let A and B be finite sets. (Recall: this means $|A|$ and $|B|$ are integers.) What are the cardinalities of the following sets?
   (a) $A \cup B$
   (b) $A \cap B$
   (c) $\mathcal{P}(A)$
   (d) $A \times B$
2. Express $A - B$ using only intersections, unions, and complements.
3. Prove De Morgan's laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
4. Prove the distributive laws for sets: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (union over intersection) and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (intersection over union). Can you prove these using the distributive laws from logic?
5. Let A_i be the set of all integers greater than i, for any integer $i \geq 0$. Describe the sets $\cup_{i=0}^{n} A_i$, $\cap_{i=0}^{n} A_i$, $\cup_{i=0}^{\infty} A_i$, and $\cap_{i=0}^{\infty} A_i$. (For the last two, the notation $\cup_{i=0}^{\infty}$ and $\cap_{i=0}^{\infty}$ means for every integer greater than or equal to 0. We won't be too worried with infinite unions and intersections in this class.)

Functions

Fix sets A and B. A function from A to B is an assignment of a unique element of B to each element of A, and we write this as $f : A \to B$ for the function and $f(a) = b$ for an evaluation of the function at an element $a \in A$.

Definitions

1. A is called the domain of f
2. B is called the codomain of f
3. If $X \subseteq A$, then $f(X) \subseteq B$ is called the image of X
4. The range of f is the image of A, $f(A) \subseteq B$.
5. The preimage of $Y \subseteq B$ is $\{a \in A \mid f(a) \in Y\}$.
6. f is injective (or one-to-one) if $f(a_1) = f(a_2) \implies a_1 = a_2$ for any $a_1, a_2 \in A$.
7. f is surjective (or onto) if for every $b \in B$, there is some $a \in A$ with $f(a) = b$.

Exercises

1. If $f : A \to B$ is surjective, what is the relationship between the range and the codomain?
2. For a function $f : A \to B$, what kind of object is the preimage of an element $b \in B$?
3. Let $A_1, A_2 \subseteq A$ for a function $f : A \to B$. Is $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$? What about $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$? (If yes, find a proof; if no, find a counterexample.) If either is false, is there some condition which makes them true?
4. Describe how $f : A \to B$ can be thought of as a subset of $A \times B$.
5. Let $B_1, B_2 \subseteq B$ for a function $f : A \to B$. Is the preimage of $B_1 \cup B_2$ equal to the union of the preimages of B_1 and B_2? If the preimage of $B_1 \cap B_2$ equal to the intersection of the preimages of B_1 and B_2? (If yes, find a proof; if no, find a counterexample.) If either is false, is there some condition which makes them true?
6. Decide whether the following are functions. If they are, classify them as surjective, injective, both, or neither:
   (a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x + 1$
   (b) $f : \mathbb{Z} \to \mathbb{Z}_{>0}$ defined by $f(x) = 2|x|$
   (c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$
   (d) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = \frac{1}{x}$.
   (e) $f : \mathbb{R} \to \mathbb{R}_{>0}$ defined by $x \to e^x$ ($\mathbb{R}_{>0}$ is the set of positive real numbers)
   (f) $f : \mathbb{R} \to \mathbb{R}$ defined by $x \to \log(x)$