Conditionals and Biconditionals

A *conditional statement* is of the form "if p, then q," and this is written as $p \to q$. A *biconditional statement* is of the form "p if and only if q," and this is written as $p \leftrightarrow q$. For a condtional statement $p \to q$, the *converse* is $q \to p$, the *contrapositive* is $\neg q \to \neg p$, and the *inverse* is $\neg p \to \neg q$.

p	q	$p \to q$	$p \leftrightarrow q$	$\neg q \to \neg p$	$\neg p \vee q$	$(\neg p \wedge \neg q) \vee (q \wedge p)$
T	T	T	T	T	T	T
T	F	F	F	F	F	F
F	T	T	F	T	T	F
F	F	T	T	T	T	T

1. Fill in the missing columns in the truth table above.
2. What is the contrapositive of the converse?

   The contrapositive of the converse is the inverse.

Logical Equivalences

Two propositions a and b are *logically equivalent* if $a \leftrightarrow b$ is always true (i.e. a and b always have the same truth value), and this is written as $a \equiv b$. A statement that is always true is a *tautology* and a statement that is always false is a *contradiction*.

1. In the truth table above, which statements are logically equivalent?

   $p \to q, \neg q \to \neg p, \neg p \vee q$ are logically equivalent and $p \leftrightarrow q, (\neg p \wedge \neg q) \vee (p \wedge q)$ are logically equivalent.

2. Give an alternative proof, without using truth tables, that $p \leftrightarrow q \equiv (\neg p \wedge \neg q) \vee (p \wedge q)$.

   $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\neg p \vee q) \wedge (\neg q \vee p) \equiv (\neg p \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge \neg q) \vee (q \wedge p)$
   $\equiv (\neg p \wedge \neg q) \vee (q \wedge p)$.

3. Label the following as tautologies, contradictions, or neither.

   (a) $p \vee \neg(p \wedge q)$
   Tautology

   (b) $(p \to q) \leftrightarrow (\neg p \to \neg q)$
   Neither

   (c) $(p \vee \neg(p \wedge q)) \to (r \wedge \neg r)$
   Contradiction

   (d) $((a \vee b) \wedge (a \to c) \wedge (b \to c)) \to c$
   Tautology

De Morgan's Laws

De Morgan's laws are logical equivalences between the negation of a conjunction (resp. disjunction) and the disjunction (resp. conjunction) of the negations. In other words, they are $\neg(p \wedge q) \equiv \neg p \vee \neg q$ and $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

1. State de Morgan's laws in English. (If it helps, pick concrete propositions for p and q.)

   $\neg(p \wedge q) \equiv \neg p \vee \neg q$: It is not the case that p and q if and only if it is not the case that p or it is not the case that q.

   $\neg(p \vee q) \equiv \neg p \wedge \neg q$: It is not the case that p or v if and only if it is not the case that q and it is not the case that q.

Quantifiers

Quantifiers are: $\forall$ and $\exists$. For a propositional function $P(x)$, $\forall x P(x)$ means that $P(x)$ is true for all x in the domain, and $\exists x P(x)$ means that $P(x)$ is true for some x in the domain. Always know what the domain is – this affects the truth value of statements involving quantifiers. For example, let $P(x)$ be the statement $x^2 = -1$. If the domain is $\mathbb{R}, \mathbb{Q}$, or $\mathbb{Z}$, then the statement $\exists x P(x)$ is false. However, if the domain is $\mathbb{C}$, then $\exists x P(x)$ is true.

1. Let $P(x, y)$ be the propositional function: $x < y$. Use $P(x, y)$ and quantifiers to express the statement: "there is no smallest real number." What is the domain for each quantifier you use?

   $\forall y \exists x P(x, y)$ Each quantifier has domain $\mathbb{R}$ here.

2. Let $P(x, y)$ be as above. Let $Q(y)$ be the propositional function $\forall x P(x, y)$ where the domain of x is $\mathbb{Z}$ (the integers). For what y is $Q(y)$ true?

   $Q(y)$ is true when $y = \infty$.

3. Let $P(x, y)$ be the propositional function: x is divisible by y. Use $P(x, y)$ and quantifiers to express the propositional function: "x is a prime number."

   A prime number is divisible by only 1 and itself (and 1 is not prime). One way express the propositional function "x is prime" is $\forall y(P(x, y) \to ((y = 1) \oplus (y = x)))$. We need the exclusive or here because otherwise we will consider 1 as a prime number.

De Morgan

We also have De Morgan's laws for quantifiers. They are: $\neg\exists x P(x) \equiv \forall x \neg P(x)$ and equivalently $\neg\forall x P(x) \equiv \exists x \neg P(x)$.

1. State these laws in English. (If it helps, pick a concrete function $P(x)$ and domain.)

   $\neg\exists x P(x) \equiv \forall x \neg P(x)$: It is not the case that there exists an x such that $P(x)$ is true if and only if for all x, $P(x)$ is false.

   $\neg\forall x P(x) \equiv \exists x \neg P(x)$: If is not the case that $P(x)$ is true for all x if and only if there exists an x such that $P(x)$ is false.