

Conditionals and Biconditionals

A *conditional statement* is of the form “if p , then q ,” and this is written as $p \rightarrow q$. A *biconditional statement* is of the form “ p if and only if q ,” and this is written as $p \leftrightarrow q$. For a conditional statement $p \rightarrow q$, the *converse* is $q \rightarrow p$, the *contrapositive* is $\neg q \rightarrow \neg p$, and the *inverse* is $\neg p \rightarrow \neg q$.

p	q	$p \rightarrow q$	$p \leftrightarrow q$	$\neg q \rightarrow \neg p$	$\neg p \vee q$	$(\neg p \wedge \neg q) \vee (q \wedge p)$
T	T	T	T			
T	F	F	F			
F	T	T	F			
F	F	T	T			

1. Fill in the missing columns in the truth table above.
2. What is the contrapositive of the converse?

Logical Equivalences

Two propositions a and b are *logically equivalent* if $a \leftrightarrow b$ is always true (i.e. a and b always have the same truth value), and this is written as $a \equiv b$. A statement that is always true is a *tautology* and a statement that is always false is a *contradiction*.

1. In the truth table above, which statements are logically equivalent?
2. Give an alternative proof, without using truth tables, that $p \leftrightarrow q \equiv (\neg p \wedge \neg q) \vee (p \wedge q)$.
3. Label the following as tautologies, contradictions, or neither.
 - (a) $p \vee \neg(p \wedge q)$
 - (b) $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$
 - (c) $(p \vee q) \rightarrow \neg r$
 - (d) $((a \vee b) \wedge (a \rightarrow c) \wedge (b \rightarrow c)) \rightarrow c$

De Morgan's Laws

De Morgan's laws are logical equivalences between the negation of a conjunction (resp. disjunction) and the disjunction (resp. conjunction) of the negations. In other words, they are $\neg(p \wedge q) \equiv \neg p \vee \neg q$ and $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

1. State de Morgan's laws in English. (If it helps, pick concrete propositions for p and q .)

Quantifiers

Quantifiers are: \forall and \exists . For a propositional function $P(x)$, $\forall xP(x)$ means that $P(x)$ is true for all x in the domain, and $\exists xP(x)$ means that $P(x)$ is true for some x in the domain. Always know what the domain is – this affects the truth value of statements involving quantifiers. For example, let $P(x)$ be the statement $x^2 = -1$. If the domain is \mathbb{R} , \mathbb{Q} , or \mathbb{Z} , then the statement $\exists xP(x)$ is false. However, if the domain is \mathbb{C} , then $\exists xP(x)$ is true.

1. Let $P(x, y)$ be the propositional function: $x < y$. Use $P(x, y)$ and quantifiers to express the statement: “there is no smallest real number.” What is the domain for each quantifier you use?
2. Let $P(x, y)$ be as above. Let $Q(y)$ be the propositional function $\forall xP(x, y)$ where the domain of x is \mathbb{Z} (the integers). For what values of y is $Q(y)$ true?
3. Let $P(x, y)$ be the propositional function: x is divisible by y . Use $P(x, y)$ and quantifiers to express the proposition “ x is a prime number.”

De Morgan

We also have De Morgan's laws for quantifiers. They are: $\neg\exists xP(x) \equiv \forall x\neg P(x)$ and equivalently $\neg\forall xP(x) \equiv \exists x\neg P(x)$.

1. State these laws in English. (If it helps, pick a concrete function $P(x)$ and domain.)